MATH 106

Mid-term Exam 1443 (First Semester)

1. Use Simpson's rule with $n = 4$ to approximate the integral $\int_1^3 \sqrt{1+x^2}\,dx$.

Solution. $\delta x = \frac{3-1}{4} = 0.5$
$x_0 = 1$
$x_1 = 1+0.5 = 1.5$
$x_2 = 1+2(0.5) = 2$
$x_3 = 1+3(0.5) = 2.5$
$x_4 = 1+4(0.5) = 3$

x_k	$f(x_k)$	m_k	$m_k f(x_k)$
1	1.4142	1	2
1.5	1.8028	4	7.2112
2	2.2361	2	4.4722
2.5	2.6926	4	10.7704
3	3.1623	1	3.1623
Sum = $\sum_{k=1}^{4} m_k f(x_k)$			27.0302

$$\int_1^3 \sqrt{x^2+1}\,dx \approx \tfrac{2}{12}\left[27.0302\right] = 4.5050.$$

2. Evaluate the integral $\int \frac{(1-\frac{1}{x^2})^5}{x^3}\,dx$.

Solution. $u = 1-\frac{1}{x^2} \Rightarrow du = \frac{2}{x^3}dx$. By substitution,

$$\frac{1}{2}\int u^5\,du = \frac{1}{2}\frac{u^6}{6}, \text{ so } \int \frac{(1-\frac{1}{x^2})^5}{x^3}\,dx = \frac{1}{12}\,(1-\frac{1}{x^2})^6 + c.$$

3. Find $\frac{dy}{dx}$ if $y = \sqrt{x}.\sqrt[3]{x+2}.\sqrt[5]{x-1}$.

Solution.

$$\ln y = \frac{1}{2}\ln x + \frac{1}{3}\ln(x+2) + \frac{1}{5}\ln(x-1)$$
$$\Rightarrow \frac{y'}{y} = \frac{1}{2x} + \frac{1}{3(x+2)+\frac{1}{5(x-1)}}$$
$$\Rightarrow y' = \left(\frac{1}{2x} + \frac{1}{3(x+2)+\frac{1}{5(x-1)}}\right)\sqrt{x}.\sqrt[3]{x+2}.\sqrt[5]{x-1}.$$

4. Evaluate the integral $\int \frac{(\sec x)^2}{\sqrt{4-(\tan x)^2}}\,dx$.

Solution. $u = \tan x \Rightarrow du = \sec^2 x dx$. By substitution, $\int \frac{1}{\sqrt{4-u^2}}\,du = \sin^{-1}(\frac{u}{2}) + c$. So, $\int \frac{(\sec x)^2}{\sqrt{4-(\tan x)^2}}\,dx = \sin^{-1}(\frac{\tan x}{2}) + c$.

5. Compute the integral $\int \frac{dx}{\sqrt{e^{2x}-1}\,dx}$.

Solution. $u = e^x \Rightarrow du = e^x dx$. By substitution, $\int \frac{1}{u\,\sqrt{u^2-1}}\,du = \sec^{-1}(u)+c$. So, $\int \frac{dx}{\sqrt{e^{2x}-1}}\,dx = \sec^{-1}(e^x)+c$.

6. Find the indefinite integral $\int \frac{dx}{x\,\sqrt{1-x^5}}\,dx$.

Solution. $\int \frac{dx}{x\,\sqrt{1-x^5}}\,dx = \int \frac{dx}{x\,\sqrt{1-(x^{\frac{5}{2}})^2}}\,dx$.

Let $u = x^{\frac{5}{2}} \Rightarrow du = \frac{5}{2}\,x^{\frac{3}{2}}\,dx$. By substitution, $\frac{2}{3}\int \frac{1}{u\,\sqrt{1-u^2}}\,du = -\frac{2}{3}\;^{-1}(u)+c$. So, $\int \frac{dx}{x\,\sqrt{1-x^5}}\,dx = -\frac{2}{3}\,sech^{-1}(x^{\frac{5}{2}})+c$.

7. Compute $\lim_{x\to 0} \frac{\cos x - 1 + x\frac{x^2}{2}}{x^4}$.

Solution. We have the indeterminate form $\frac{0}{0}$. By applying L'Hôpital's rule, we have

$$\lim_{x\to 0} \frac{-\sin x + x}{4x^3} \qquad \text{(apply L'Hôpital's rule)}$$

$$\lim_{x\to 0} \frac{-\cos x + 1}{12x^2} \qquad \text{(apply L'Hôpital's rule)}$$

$$\lim_{x\to 0} \frac{\sin x + x}{24x} \qquad \text{(apply L'Hôpital's rule)}$$

$$\lim_{x\to 0} \frac{\cos x + x}{24} = \frac{1}{24}\ .$$

8. Integrate by parts twice to compute $\int (\ln x)^2\,dx$.

Solution. Let $I = \int (\ln x)^2\,dx$, then

$$u = (\ln x)^2 \Rightarrow du = \frac{2\ln x}{x}\,dx\ ,$$

$$dv = dx \Rightarrow v = \int 1\,dx = x.$$

$I = x(\ln x)^2 - 2\int \ln x\,dx$

Let $J = \int \ln x\,dx$, then

$$u = \ln x \Rightarrow du = \frac{1}{x}\,dx\ ,$$

$$dv = dx \Rightarrow v = \int 1\,dx = x.$$

$J = x\ln x - x \Rightarrow I = x(x\ln x)^2 - 2\,(x\ln x - x) + c.$

9. Find $\int (\tan x)^5\ (\sec\ x)^3\ dx.$

Solution.

$$\begin{aligned}\int (\tan x)^5\ (\sec\ x)^3\ dx &= \int (\tan x)^4\ (\sec\ x)^2\ \sec x\ \tan x\ dx\\ &= \int (\sec^2 x - 1)^2\ (\sec\ x)^2\ \sec x\ \tan x\ dx\\ &= \frac{(\sec x)^7}{7} - \frac{2(\sec x)^5}{5} - \frac{(\sec x)^3}{3} + c.\end{aligned}$$

10. Evaluate the integral $\int \frac{x^2}{\sqrt{9-x^2}}\ dx.$

Solution. Let $x = 3\sin\theta \Rightarrow dx = 3cos\theta$. By substitution,

$\int \frac{9\sin^2\theta}{3\ cos\theta}\ 3\cos\theta\ d\theta = \frac{9}{2}\int (1 - cos2\theta)\ d\theta = \frac{9}{2}(\theta - \sin\theta\cos\theta) + c.$ By substitution,

$$\int \frac{x^2}{\sqrt{9-x^2}}\ dx = \frac{9}{2}(\sin^{-1}\frac{x}{3} - \frac{x\sqrt{9-x^2}}{9}) + c\ .$$

11. Compute the indefinite integral $\int \frac{x^2+8x+10}{x^2+6x+11}\ dx.$

Solution.

$\int \frac{x^2+8x+10}{x^2+6x+11}\ dx = \int 1\ dx + \int \frac{2x-1}{(x-3)^2+2}\ dx = x + \ln|(x+3)^2+2| - 7\tan^{-1}(\frac{x+3}{2}) + c.$