7.4 Paired Comparisons:

- In this section, we are interested in comparing the means of two related (non-independent/dependent) normal populations.
- In other words, we wish to make statistical inference for the difference between the means of two related normal populations.
- Paired t-Test concerns about testing the equality of the means of two related normal populations.

Examples of related populations are:

1. Height of the father and height of his son.
2. Mark of the student in MATH and his mark in STAT.
3. Pulse rate of the patient before and after the medical treatment.
4. Hemoglobin level of the patient before and after the medical treatment.

Example: (effectiveness of a diet program)

Suppose that we are interested in studying the effectiveness of a certain diet program. Let the random variables X and Y are as follows:

X = the weight of the individual before the diet program

Y= the weight of the same individual after the diet program

We assume that the distributions of these random variables are normal with means μ_1 and μ_2, respectively.

These two variables are related (dependent/non-independent) because they are measured on the same individual.

Populations:

1-st population (X): weights before a diet program

mean = μ_1

2-nd population (Y): weights after the diet program

mean = μ_2

Question:

Does the diet program have an effect on the weight?

Answer is:

No if $\mu_1 = \mu_2$ $(\mu_1 - \mu_2 = 0)$

Yes if $\mu_1 \neq \mu_2$ $(\mu_1 - \mu_2 \neq 0)$

Therefore, we need to test the following hypotheses:

Hypotheses:

H_o: $\mu_1 = \mu_2$ (H_o: the diet program has no effect on weight)

H_A: $\mu_1 \neq \mu_2$ (H_A: the diet program has an effect on weight)

Equivalently we may test:

H_o: $\mu_1 - \mu_2 = 0$

H_A: $\mu_1 - \mu_2 \neq 0$

Testing procedures:

- We select a random sample of n individuals. At the beginning of the study, we record the individuals' weights before the diet program (X). At the end of the diet program, we record the individuals' weights after the program (Y). We end up with the following information and calculations:

Individual i	Weight before X_i	Weight after Y_i	Difference $D_i = X_i - Y_i$
1	X_1	Y_1	$D_1 = X_1 - Y_1$
2	X_2	Y_2	$D_2 = X_2 - Y_2$
. .	. .	. .	
Individual i	Weight before X_i	Weight after Y_i	Difference $D_i = X_i - Y_i$
.	.	.	
n	X_n	Y_n	$D_n = X_n - Y_n$

- Hypotheses:

  H_o: the diet program has no effect on weight

  H_A: the diet program has an effect on weight

Equivalently,

H_o: $\mu_1 = \mu_2$

H_A: $\mu_1 \neq \mu_2$

Equivalently,

H_o: $\mu_1 - \mu_2 = 0$

H_A: $\mu_1 - \mu_2 \neq 0$

Equivalently,

H_o: $\mu_D = 0$

H_A: $\mu_D \neq 0$

where:

$$\mu_D = \mu_1 - \mu_2$$

- We calculate the following quantities:
  - The differences (D-observations):

$$D_i = X_i - Y_i \quad (i=1, 2, \ldots, n)$$

  - Sample mean of the D-observations (differences):

$$\bar{D} = \frac{\sum_{i=1}^{n} D_i}{n} = \frac{D_1 + D_2 + \ldots + D_n}{n}$$

  - Sample variance of the D-observations (differences):

$$S_D^2 = \frac{\sum_{i=1}^{n} \left(D_i - \bar{D}\right)^2}{n-1} = \frac{\left(D_1 - \bar{D}\right)^2 + \left(D_2 - \bar{D}\right)^2 + \cdots + \left(D_n - \bar{D}\right)^2}{n-1}$$

  - Sample standard deviation of the D-observations:

$$S_D = \sqrt{S_D^2}$$

- Test Statistic:

  We calculate the value of the following test statistic:

$$t = \frac{\overline{D}}{S_D / \sqrt{n}} \sim t(n-1)$$

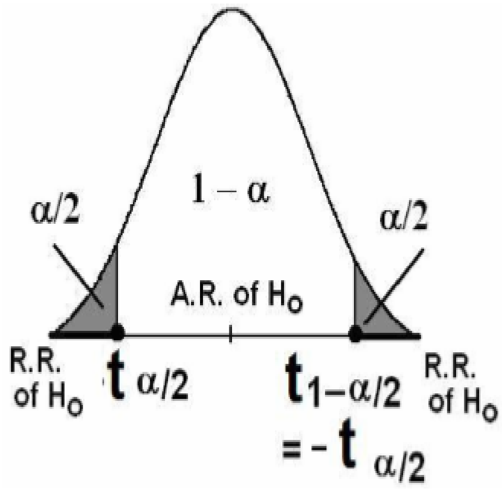
  This statistic has a t-distribution with df = ν = n−1.

- Rejection Region of H_o:

  Critical values are: $t_{\alpha/2}$ and $t_{1-\alpha/2} = -t_{\alpha/2}$.

  The rejection region (critical region) at the significance level α is:

$$t < t_{\alpha/2} \text{ or } t > t_{1-\alpha/2} = -t_{\alpha/2}$$



- Decision:

We reject H_o and accept H_A at the significance level α if $T \in R.R.$,i.e., if:

$$t < t_{\alpha/2} \quad \text{or} \quad t > t_{1-\alpha/2} = -t_{\alpha/2}$$

Numerical Example:

In the previous example, suppose that the sample size was 10 and the data were as follows:

Individual (i)	1	2	3	4	5	6	7	8	9	10
Weight before (X_i)	86.6	80.2	91.5	80.6	82.3	81.9	88.4	85.3	83.1	82.1
Weight after (Y_i)	79.7	85.9	81.7	82.5	77.9	85.8	81.3	74.7	68.3	69.7

Does these data provide sufficient evidence to allow us to conclude that the diet program is effective? Use α=0.05 and assume that the populations are normal.

Solution:

μ_1 = the mean of weights before the diet program

μ_2 = the mean of weights after the diet program

Hypotheses:

$H_o: \mu_1 = \mu_2$ (H_o: the diet program is not effective)

$H_A: \mu_1 \neq \mu_2$ (H_A: the diet program is effective)

Equivalently,

$H_o: \mu_D = 0$

$H_A: \mu_D \neq 0$ (where: $\mu_D = \mu_1 - \mu_2$)

Calculations:

i	X_i	Y_i	$D_i = X_i - Y_i$
1	86.6	79.7	6.9
2	80.2	85.9	-5.7
3	91.5	81.7	9.8
4	80.6	82.5	-1.9
5	82.3	77.9	4.4
6	81.9	85.8	-3.9
7	88.4	81.3	7.1
8	85.3	74.7	10.6
9	83.1	68.3	14.8
10	82.1	69.7	12.4
sum	$\sum X = 842$	$\sum Y = 787.5$	$\sum D = 54.5$

$$\bar{D} = \frac{\sum_{i=1}^{n} D_i}{n} = \frac{54.5}{10} = 5.45$$

$$S_D^2 = \frac{\sum_{i=1}^{n} (D_i - \bar{D})^2}{n-1} = \frac{(6.9-5.45)^2 + \ldots + (12.4-5.45)^2}{10-1} = 50.3283$$

$$S_D = \sqrt{S_D^2} = \sqrt{50.3283} = 7.09$$

Test Statistic:

$$t = \frac{\bar{D}}{S_D / \sqrt{n}} = \frac{5.45}{7.09/\sqrt{10}} = 2.431$$

Degrees of freedom:

$$df = \nu = n-1 = 10-1 = 9$$

Significance level: $\alpha = 0.05$

Rejection Region of H_o:

Critical values: $t_{0.025} = -2.262$ and $t_{0.975} = -t_{0.025} = 2.262$

Critical Region: $t < -2.262$ or $t > 2.262$

Decision:

Since $t= 2.43 \in R.R.$, i.e., $t=2.43 > t_{0.975}=-t_{0.025}=2.262$, we reject:

$$H_o: \mu_1 = \mu_2 \quad \text{(the diet program is not effective)}$$

and we accept:

$$H_1: \mu_1 \neq \mu_2 \quad \text{(the diet program is effective)}$$

Consequently, we conclude that the diet program is effective at $\alpha=0.05$.

Note:

- The sample mean of the weights before the program is $\bar{X} = 84.2$
- The sample mean of the weights after the program is $\bar{Y} = 78.75$
- Since the diet program is effective and since $\bar{X} = 84.2 > \bar{Y} = 78.75$, we can conclude that the program is effective in reducing the weight.

Confidence Interval for the Difference between the Means of Two Related Normal Populations ($\mu_D = \mu_1 - \mu_2$):

In this section, we consider constructing a confidence interval for the difference between the means of two related (non-independent) normal populations. As before, let us define the difference between the two means as follows:

$$\mu_D = \mu_1 - \mu_2$$

where μ_1 is the mean of the first population and μ_2 is the mean of the second population. We assume that the two normal populations are not independent.

Result:

A $(1-\alpha)100\%$ confidence interval for $\mu_D = \mu_1 - \mu_2$ is:

$$\overline{D} \pm t_{1-\frac{\alpha}{2}} \frac{S_D}{\sqrt{n}}$$

$$\overline{D} - t_{1-\frac{\alpha}{2}} \frac{S_D}{\sqrt{n}} < \mu_D < \overline{D} + t_{1-\frac{\alpha}{2}} \frac{S_D}{\sqrt{n}}$$

where:

$$\overline{D} = \frac{\sum_{i=1}^{n} D_i}{n} \quad , \quad S_D = \sqrt{S_D^2} \, , \quad S_D^2 = \frac{\sum_{i=1}^{n} (D_i - \overline{D})^2}{n-1} \, , \quad \text{df} = \nu = \text{n} - 1.$$

Example:

Consider the data given in the previous numerical example:

Individual (i)	1	2	3	4	5	6	7	8	9	10
Weight before (X_i)	86.6	80.2	91.5	80.6	82.3	81.9	88.4	85.3	83.1	82.1
Weight after (Y_i)	79.7	85.9	81.7	82.5	77.9	85.8	81.3	74.7	68.3	69.7

Find a 95% confidence interval for the difference between the mean of weights before the diet program (μ_1) and the mean of weights after the diet program (μ_2).

Solution:

We need to find a 95% confidence interval for $\mu_D = \mu_1 - \mu_2$:

$$\bar{D} \pm t_{1-\frac{\alpha}{2}} \frac{S_D}{\sqrt{n}}$$

We have found:

$$\bar{D} = 5.45 \quad , \qquad S_D^2 = 50.3283 \quad , \quad S_D = \sqrt{S_D^2} = 7.09$$

The value of the reliability coefficient $t_{1-\frac{\alpha}{2}}$ $(df = \nu = n - 1 = 9)$ is

$t_{1-\frac{\alpha}{2}} = t_{0.975} = 2.262$.

Therefore, a 95% confidence interval for $\mu_D = \mu_1 - \mu_2$ is

$$5.45 \pm (2.262)\, \frac{7.09}{\sqrt{10}}$$

$$5.45 \pm 5.0715$$

$$0.38 < \mu_D < 10.52$$

$$0.38 < \mu_1 - \mu_2 < 10.52$$

We are 95% confident that $\mu_D = \mu_1 - \mu_2 \in (0.38, 10.52)$.

Note: Since this interval does not include 0, we say that 0 is not a candidate for the difference between the population means ($\mu_1 - \mu_2$), and we conclude that $\mu_1 - \mu_2 \neq 0$, i.e., $\mu_1 \neq \mu_2$. Thus we arrive at the same conclusion by means of a confidence interval.

7.5 Hypothesis Testing: A Single Population Proportion (p).

Recall:

- p = Population proportion of elements of Type A in the population

$$p = \frac{no.\ of\ elements\ of\ type\ A\ in\ the\ population}{Total\ no.\ of\ elements\ in\ the\ population}$$

$$p = \frac{A}{N} \qquad (N = population\ size)$$

- $.n$ = sample size
- X = no. of elements of type A in the sample of size n.
- $\hat{p}$ = Sample proportion elements of Type A in the sample

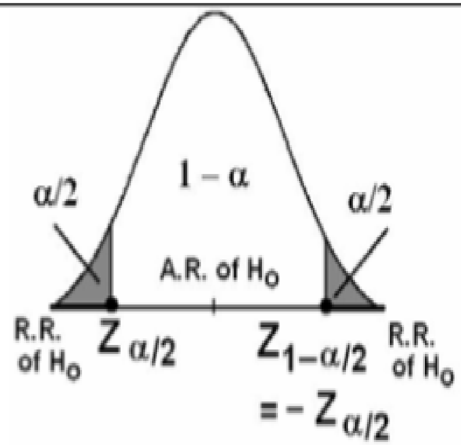
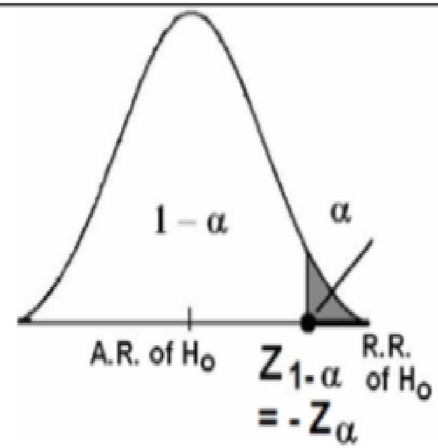
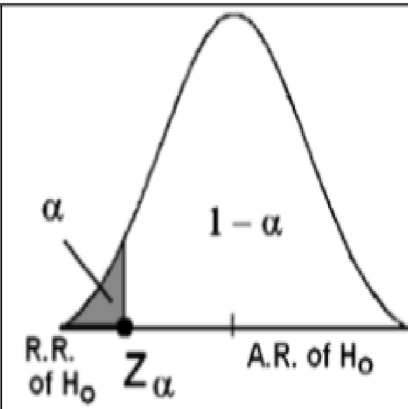
$$\hat{p} = \frac{no.\ of\ elements\ of\ type\ A\ in\ the\ sample}{no.\ of\ elements\ in\ the\ sample}$$

$$\hat{p} = \frac{X}{n}$$ (n=sample size=no. of elements in the sample)

- $\hat{p}$ is a "good" point estimate for p.
- For large n, ($n \geq 30$, $np > 5$), we have

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

- Let p_0 be a given known value.
- Test Procedure:

<table>
<tr><td>Hypotheses</td><td>$H_0: p = p_0$
$H_A: p \neq p_0$</td><td>$H_0: p \leq p_0$
$H_A: p > p_0$</td><td>$H_0: p \geq p_0$
$H_A: p < p_0$</td></tr>
<tr><td>Test Statistic (T.S.)</td><td colspan="3">$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0,1)$$</td></tr>
<tr><td>R.R. & A.R. of H_0</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td>Decision:</td><td colspan="3">Reject H_0 (and accept H_A) at the significance level α if:</td></tr>
<tr><td></td><td>$Z < Z_{\alpha/2}$ or
$Z > Z_{1-\alpha/2} = -Z_{\alpha/2}$
Two-Sided Test</td><td>$Z > Z_{1-\alpha} = -Z_{\alpha}$
One-Sided Test</td><td>$Z < Z_{\alpha}$
One-Sided Test</td></tr>
</table>

Example:

A researcher was interested in the proportion of females in the population of all patients visiting a certain clinic. The researcher claims that 70% of all patients in this population are females. Would you agree with this claim if a random survey shows that 24 out of 45 patients are females? Use a 0.10 level of significance.

Solution:

p = Proportion of female in the population.

.n=45 (large)

X= no. of female in the sample = 24

$\hat{p}$ = proportion of females in the sample

$$\hat{p} = \frac{X}{n} = \frac{24}{45} = 0.5333$$

$$p_o = \frac{70}{100} = 0.7$$

α=0.10

Hypotheses:

H_o: $p = 0.7$ (p_o=0.7)

H_A: $p \neq 0.7$

Level of significance:

α=0.10

Test Statistic (T.S.):

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

$$= \frac{0.5333 - 0.70}{\sqrt{\frac{(0.7)(0.3)}{45}}} = -2.44$$

Rejection Region of H_0 (R.R.):

Critical values:

$$Z_{\alpha/2} = Z_{0.05} = -1.645$$

$$-Z_{\alpha/2} = -Z_{0.05} = 1.645$$

We reject H_0 if:

$$Z < Z_{\alpha/2} = Z_{0.05} = -1.645$$

or

$$Z > -Z_{\alpha/2} = -Z_{0.05} = 1.645$$

Decision:

Since $Z = -2.44 \in$ Rejection Region of H_0 (R.R), we reject

$H_0: p = 0.7$ and accept $H_A: p \neq 0.7$ at $\alpha = 0.1$. Therefore, we do not agree with the claim stating that 70% of the patients in this population are females.

Example:

In a study on the fear of dental care in a certain city, a survey showed that 60 out of 200 adults said that they would hesitate to take a dental appointment due to fear. Test whether the proportion of adults in this city who hesitate to take dental appointment is less than 0.25. Use a level of significance of 0.025.

Solution:

p = Proportion of adults in the city who hesitate to take a dental appointment.

.n= 200 (large)

X= no. of adults who hesitate in the sample = 60

$\hat{p}$ = proportion of adults who hesitate in the sample

$$\hat{p} = \frac{X}{n} = \frac{60}{200} = 0.3$$

p_0=0.25

α=0.025

Hypotheses:

H_0: $p \geq 0.25$ (p_0=0.25)

H_A: $p < 0.25$ (research hypothesis)

Level of significance:

α=0.025

Test Statistic (T.S.):

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.3 - 0.25}{\sqrt{\frac{(0.25)(0.75)}{200}}} = 1.633$$

Rejection Region of H_0 (R.R.):

Critical value: $Z_{\alpha} = Z_{0.025} = -1.96$

Critical Region:

We reject H_0 if: $Z < Z_{\alpha} = Z_{0.025} = -1.96$

Decision:

Since $Z=1.633 \in$ Acceptance Region of H_o (A.R.), we accept (do not reject) H_o: $p \geq 0.25$ and we reject H_A: $p < 0.25$ at $\alpha=0.025$. Therefore, we do not agree with claim stating that the proportion of adults in this city who hesitate to take dental appointment is less than 0.25.

7.6 Hypothesis Testing: The Difference Between Two Population Proportions (p_1-p_2).

Hypotheses:

We choose one of the following situations:

(i) $H_o: p_1 = p_2$ against $H_A: p_1 \neq p_2$

(ii) $H_o: p_1 \geq p_2$ against $H_A: p_1 < p_2$

(iii) $H_o: p_1 \leq p_2$ against $H_A: p_1 > p_2$

or equivalently,

(i) $H_o: p_1 - p_2 = 0$ against $H_A: p_1 - p_2 \neq 0$

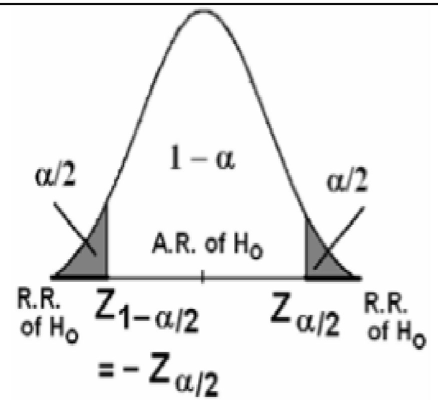
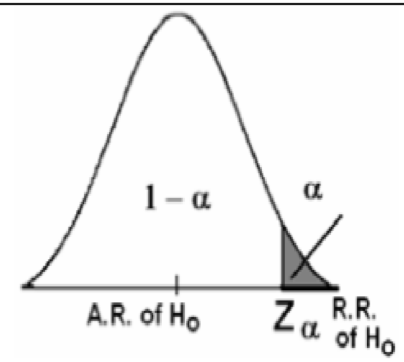
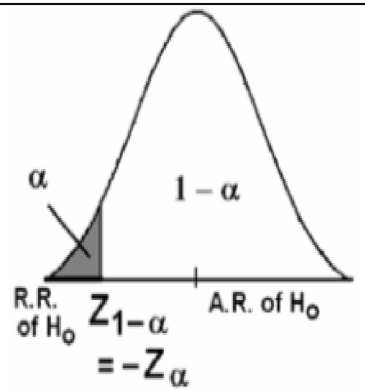
(ii) $H_o: p_1 - p_2 \geq 0$ against $H_A: p_1 - p_2 < 0$

(iii) $H_o: p_1 - p_2 \leq 0$ against $H_A: p_1 - p_2 > 0$

Note, under the assumption of the equality of the two population proportions (H_o: $p_1 = p_2 = p$), the pooled estimate of the common proportion p is:

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2} \qquad (\bar{q} = 1 - \bar{p})$$

Testing Procedure:

Hypotheses	$H_o: p_1 - p_2 = 0$ $H_A: p_1 - p_2 \neq 0$	$H_o: p_1 - p_2 \leq 0$ $H_A: p_1 - p_2 > 0$	$H_o: p_1 - p_2 \geq 0$ $H_A: p_1 - p_2 < 0$
Test Statistic (T.S.)	$$Z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\frac{\bar{p}(1-\bar{p})}{n_1} + \frac{\bar{p}(1-\bar{p})}{n_2}}} \sim N(0,1)$$		
R.R. and A.R. of H_o			
Decision:	Reject H_o (and accept H_1) at the significance level α if $Z \in$ R.R.:		
Critical Values	$Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$ Two-Sided Test	$Z > Z_{\alpha}$ One-Sided Test	$Z < -Z_{\alpha}$ One-Sided Test

Example:

In a study about the obesity (overweight), a researcher was interested in comparing the proportion of obesity between males and females. The researcher has obtained a random sample of 150 males and another independent random sample of 200 females. The following results were obtained from this study.

	n	Number of obese people
Males	150	21
Females	200	48

Can we conclude from these data that there is a difference between the proportion of obese males and proportion of obese females? Use $\alpha = 0.05$.

Solution:

Males	Females
$n_1 = 150$	$n_2 = 200$
$X_1 = 21$	$X_2 = 48$
$\hat{p}_1 = \frac{X_1}{n_1} = \frac{21}{150} = 0.14$	$\hat{p}_2 = \frac{X_2}{n_2} = \frac{48}{200} = 0.24$

The pooled estimate of the common proportion p is:

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{21 + 48}{150 + 200} = 0.197$$

Hypotheses:

$H_o: p_1 = p_2$

$H_A: p_1 \neq p_2$

or

$H_o: p_1 - p_2 = 0$

$H_A: p_1 - p_2 \neq 0$

Level of significance: $\alpha = 0.05$

Test Statistic (T.S.):

$$Z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\frac{\bar{p}(1-\bar{p})}{n_1} + \frac{\bar{p}(1-\bar{p})}{n_2}}} = \frac{(0.14 - 0.24)}{\sqrt{\frac{0.197 \times 0.803}{150} + \frac{0.197 \times 0.803}{200}}} = -2.328$$

Rejection Region (R.R.) of H_0:

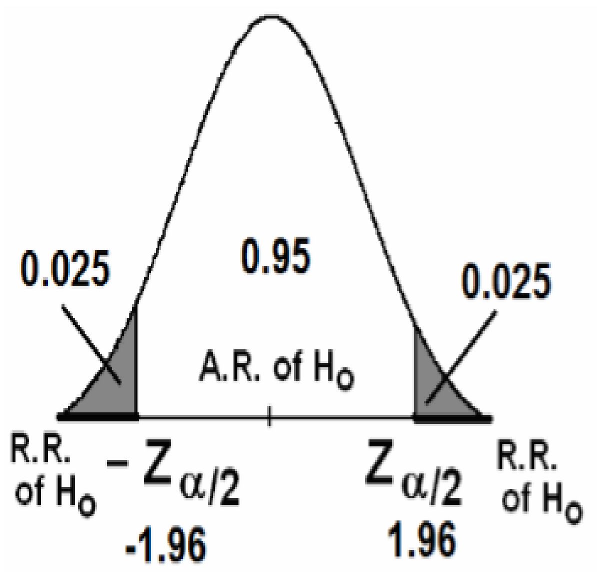
Critical values:

$Z_{\alpha/2} = Z_{0.025} = -1.96$

$Z_{1-\alpha/2} = Z_{0975} = 1.96$

Critical region:

Reject H_0 if: $Z < -1.96$ or $Z > 1.96$



Decision:

Since Z= −2.328 ∈R.R., we reject H_o: $p_1 = p_2$ and accept H_A: $p_1 \neq p_2$ at α=0.05. Therefore, we conclude that there is a difference between the proportion of obese males and the proportion of obese females. Additionally, since, $\hat{p}_1 = 0.14 < \hat{p}_2 = 0.24$, we may conclude that the proportion of obesity for females is larger than that for males.