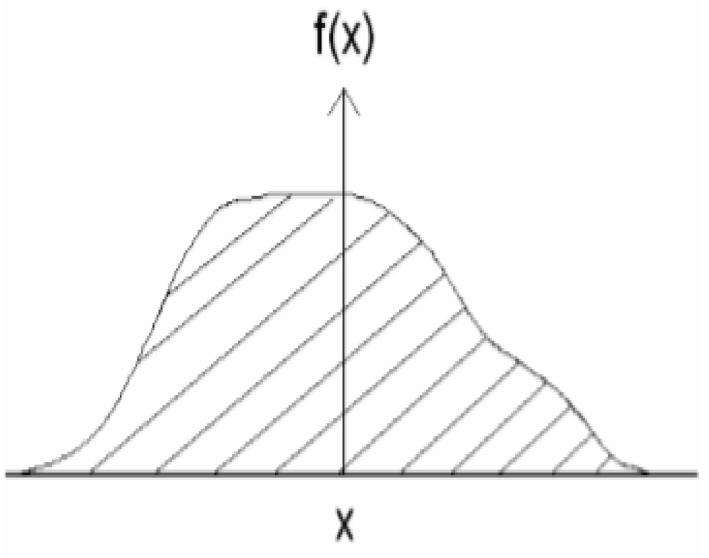
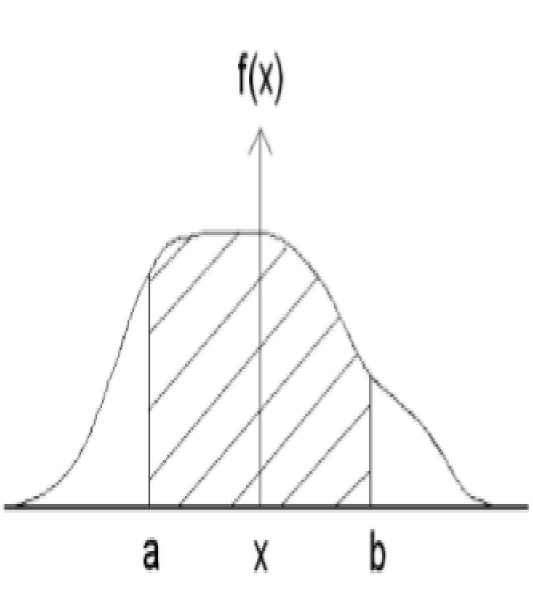
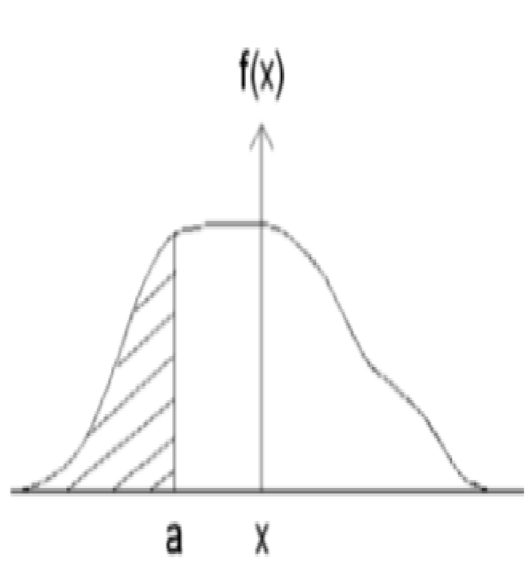
4.2. Probability Distributions of Continuous Random Variables:

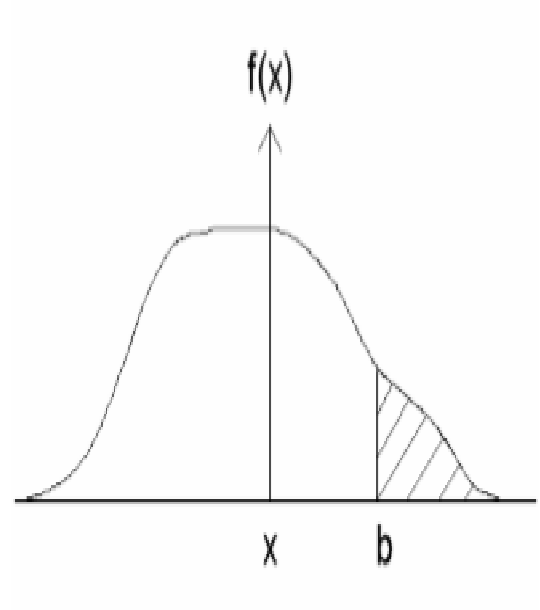
For any continuous r. v. X, there exists a function f(x), called the probability density function (pdf) of X , for which the total area under the curve of f(x) equals to 1.



$$Total\ area = \int_{-\infty}^{\infty} f(x)\,dx = 1$$







$$P(a \le X \le b) = \int_a^b f(x)\,dx = area$$

$$P(X \le a) = \int_{-\infty}^{a} f(x)\,dx = area$$

$$P(X \ge b) = \int_b^{\infty} f(x)\,dx = area$$

Note:

If X is continuous r.v. then:

1. $P(X=a)=0$ for any a.
2. $P(X \le a)=P(X<a)$
3. $P(X \ge b)=P(X>b)$
4. $P(a \le X \le b)=P(a \le X<b)=P(a<X \le b)=P(a<X<b)$
5. $P(X \le x)=$ cumulative probability
6. $P(X \ge a)=1-P(X<a)=1-P(X \le a)$
7. $P(a \le X \le b)=P(X \le b)-P(X \le a)$

4.2.1 Normal Distribution

- One of the most important continuous distributions.
- Many measurable characteristics are normally or approximately normally distributed.

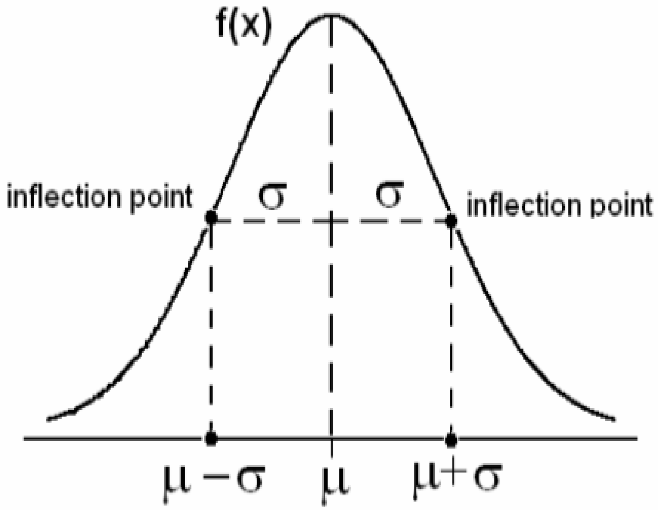
(Examples: height, weight, …)

- The probability density function of the normal distribution is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \quad ; -\infty < x < \infty$$

where (e=2.71828) and (π=3.14159).

The parameters of the distribution are the mean (μ) and the standard deviation (σ).



mean = μ

standard deviation = σ

variance = σ^2

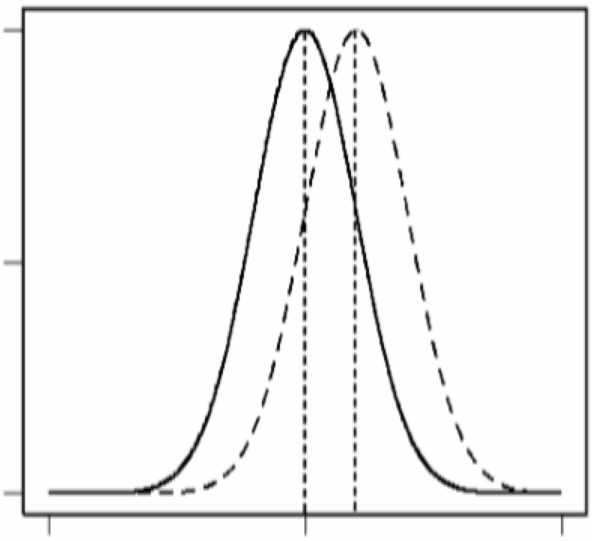
3. The highest point of the curve of f(x) at the mean μ.
   (Mode = μ)
4. The curve of f(x) is symmetric about the mean μ.

   μ = mean = mode = median
5. The normal distribution depends on two parameters:

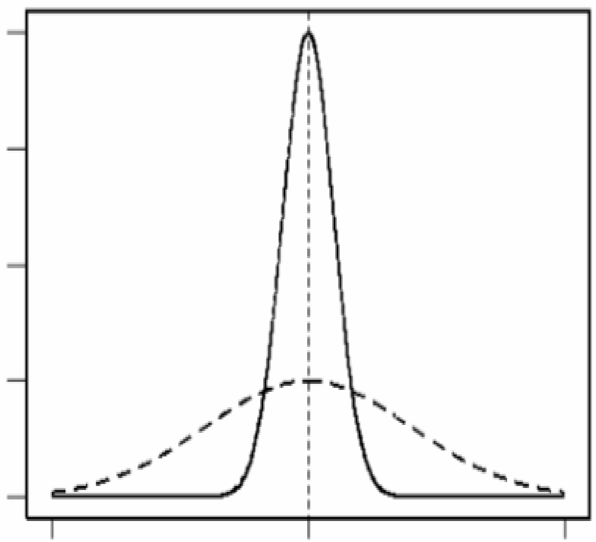
   mean = μ (determines the location)

   standard deviation = σ (determines the shape)
6. If the r.v. X is normally distributed with mean μ and standard deviation σ (variance σ^2), we write:

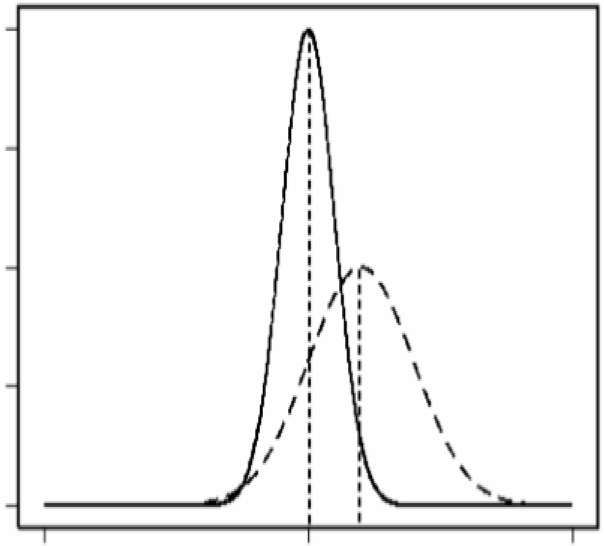
   $X \sim \text{Normal}(\mu, \sigma^2)$ or $X \sim N(\mu, \sigma^2)$



$\mu_1 < \mu_2, \sigma_1 = \sigma_2$



$\mu_1 = \mu_2, \sigma_1 < \sigma_2$



$\mu_1 < \mu_2, \sigma_1 < \sigma_2$

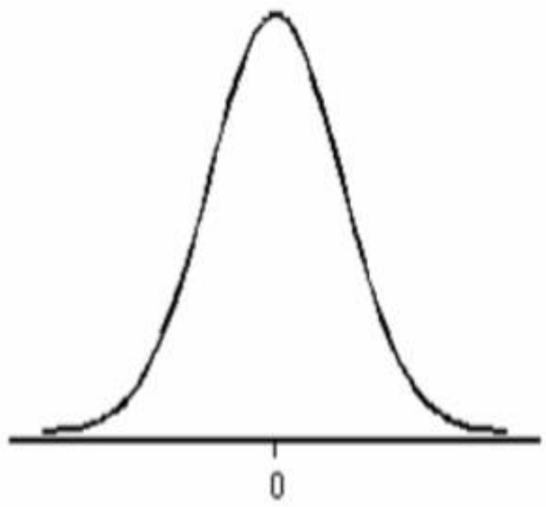
4.2.2 Standard Normal Distribution

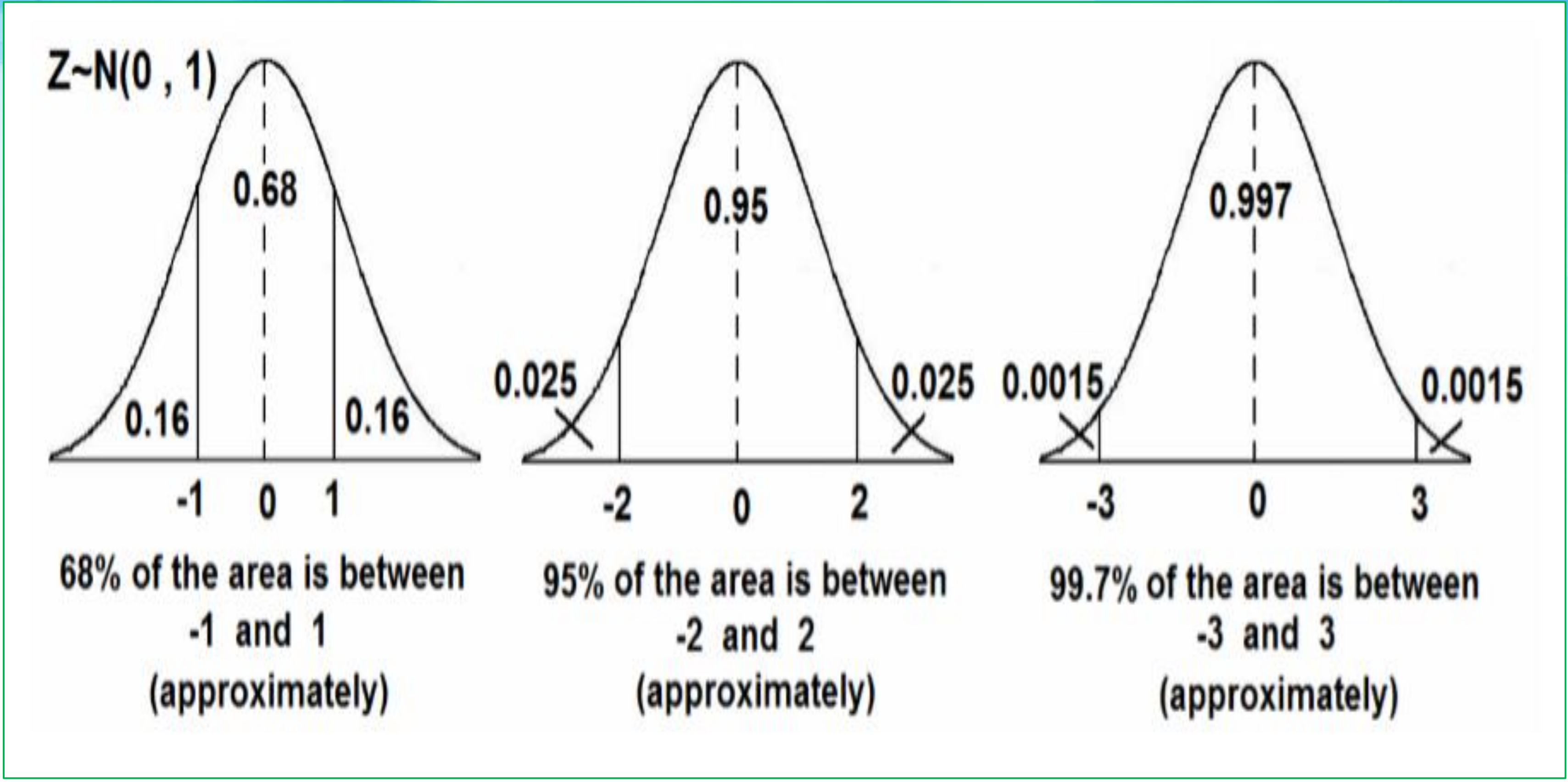
The standard normal random variable is denoted by (Z), and we write:

$$Z \sim N(0, 1)$$

The probability density function (pdf) of $Z \sim N(0,1)$ is given by:

$$f(z) = n(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$





Result:

If $X \sim \text{Normal}\left(\mu, \sigma^2\right)$, then $Z = \dfrac{X-\mu}{\sigma} \sim \text{Normal}\,(0,1)$.

4.2.3 Calculating Probabilities of Normal (0,1):

Suppose $Z \sim Normal\ (0,1)$.

(i) $P(Z \leq a) =$ From the table

(ii) $P(Z \geq b) = 1 - P(Z \leq b)$

Where:

$P(Z \leq b)$ = From the table

(iii) $P(a \leq Z \leq b) = P(Z \leq b) - P(z \leq a)$

Where:

$P(Z \leq b)$ = from the table

$P(z \leq a)$ = from the table

(iv) $P(Z = a) = 0$ for every a.

Finding $P(Z \leq z)$ from the table

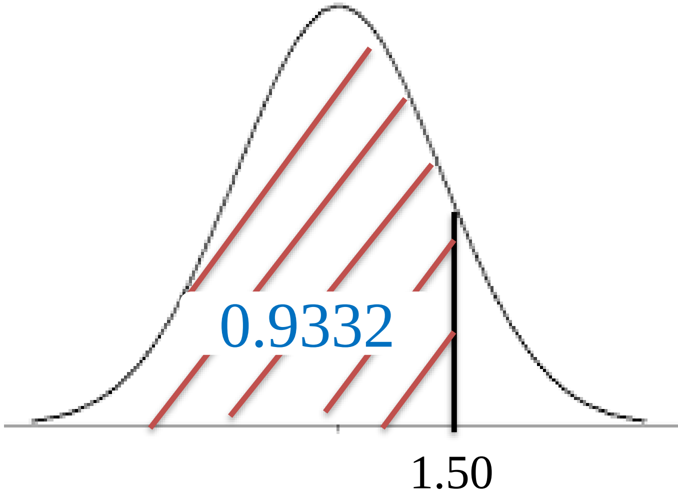
consider that the value of z is rounded to 2 decimal places as $z = a.bc$

z	0.00	0.01	z	……..	$0.0c$	……..	0.09
-3.4							
-3.3							
⋮							
$a.b$					$P(Z \leq a.bc)$		
⋮							
3.4							

Example(1): $Z \sim N(0,1)$

1) $P(Z \leq 1.50)$

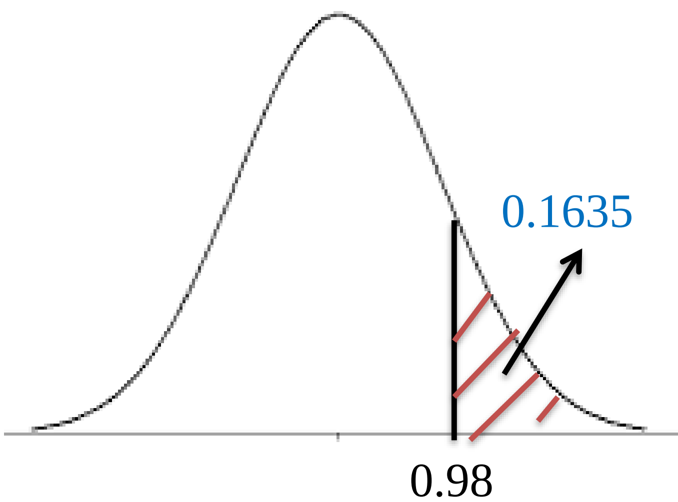
z	0.00	0.01	0.02	…….	0.09
-3.4					
-3.3					
⋮					
1.5 →	0.9332				
⋮					
3.4					



Example(2): $Z \sim N(0,1)$

2) $P(Z \geq 0.98) = 1 - P(Z \leq 0.98) = 1 - 0.8365 = 0.1635$

z	0.00	0.01	....	0.08	0.09
-3.4					
-3.3					
⋮					
0.9				0.8365	
⋮					
3.4					

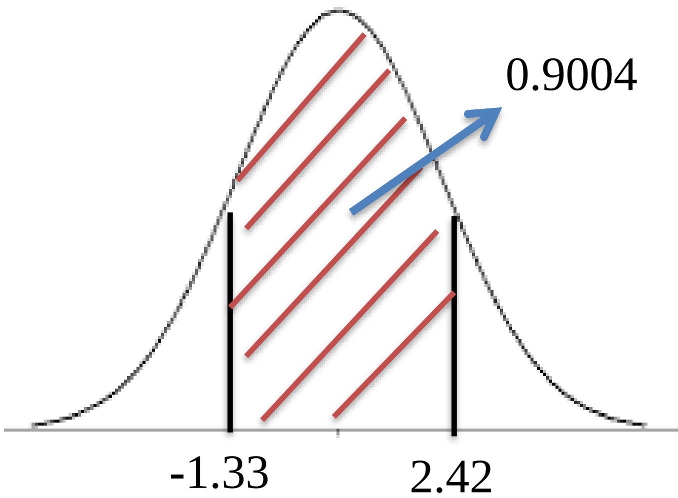


Example (3): $Z \sim N(0,1)$

3) $P(-1.33 \leq Z \leq 2.42) = P(Z \leq 2.42) - P(Z \leq -1.33)$

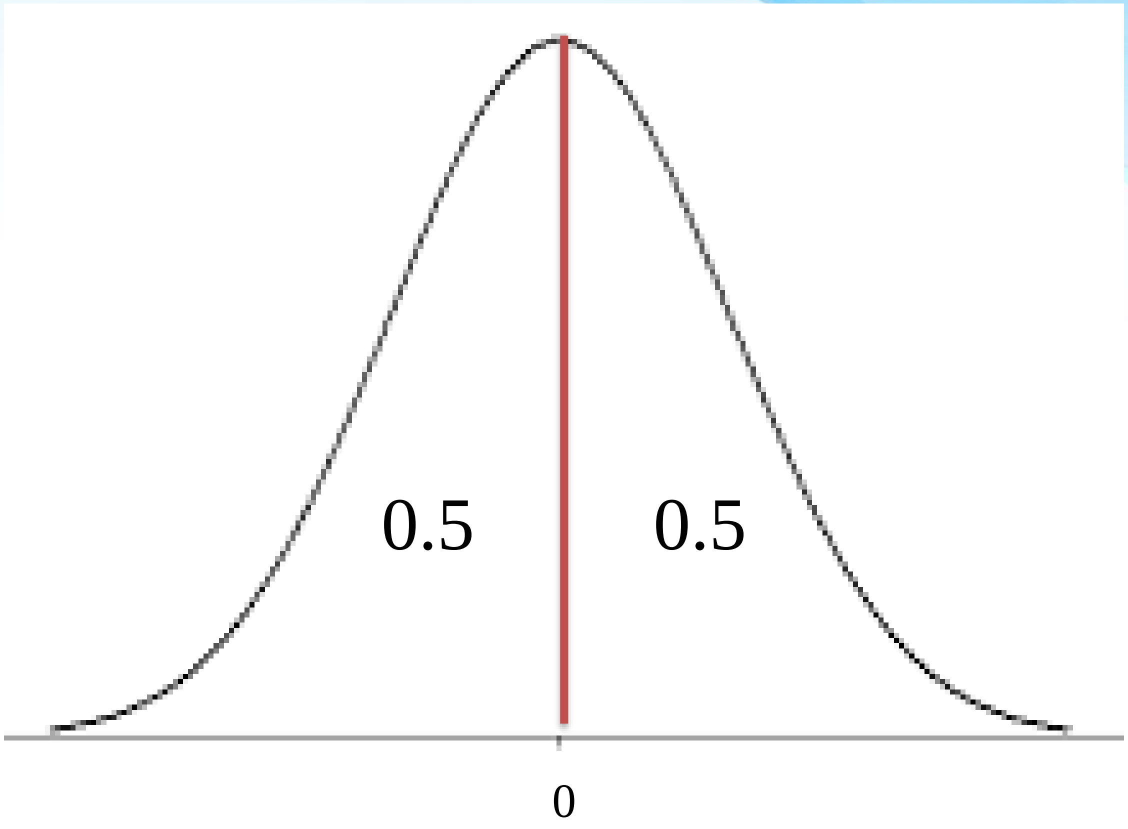
$= 0.9922 - 0.0918 = 0.9004$

z	0.00	0.01	0.02	0.03	.....
⋮					
-1.3				0.0918	
⋮					
2.4			0.9922		
⋮					



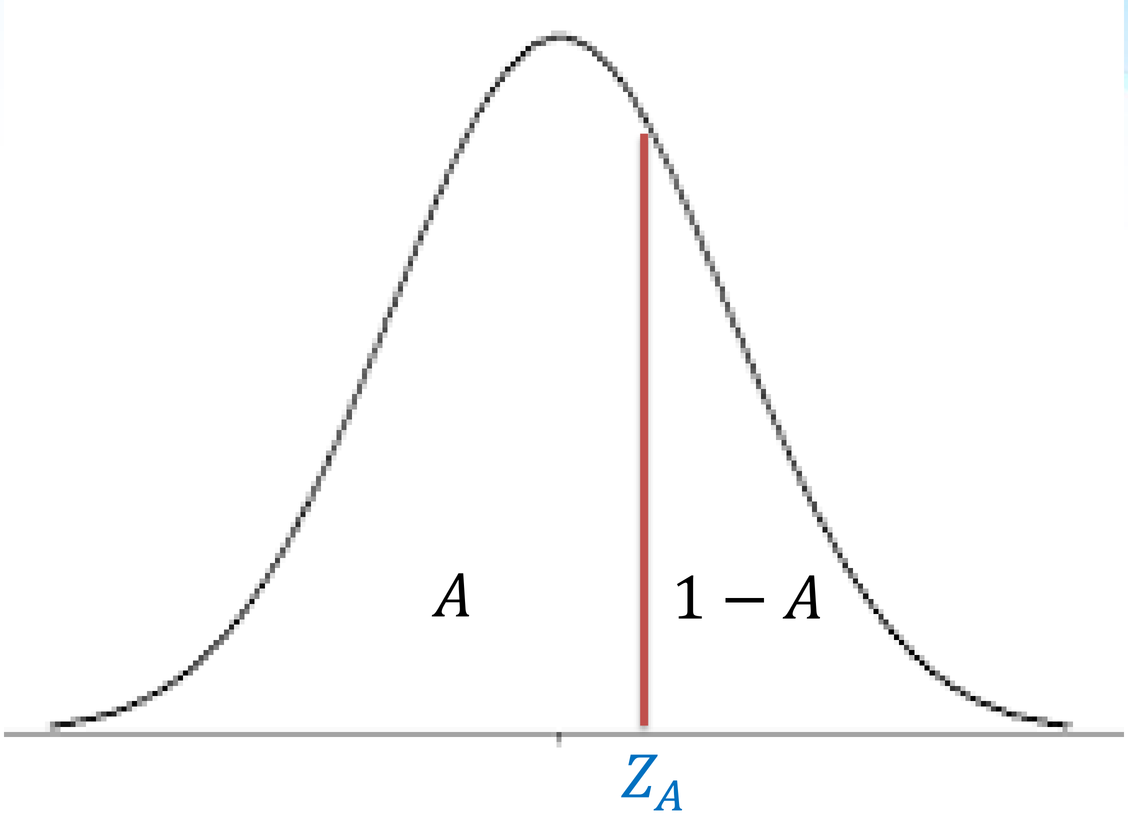
Example(4): $Z \sim N(0,1)$

4) $P(Z \geq 0) = P(Z \leq 0) = 0.5$



Notation

$$P(Z \leq Z_A) = A$$



Example: Z ~ N(0,1)

If $P(Z \le a) = 0.9505$

Z	$\ldots$	0.05	$\ldots$
$\vdots$		$\Uparrow$	
1.6	$\Leftarrow$	0.9505	
$\vdots$			

Then $a = 1.65$

Example: $Z \sim N(0,1)$

$$Z_{0.90} = 1.285$$

$$Z_{0.95} = 1.645$$

$$Z_{0.975} = 1.96$$

$$Z_{0.99} = 2.325$$

Exercise

If $P(Z \geq k) = 0.0207$

Find k?

What do you see?

- **Calculating Probabilities of Normal (μ, σ^2):**

$$X \sim \text{Normal}\left(\mu, \sigma^2\right) \quad \Leftrightarrow \quad Z = \frac{X-\mu}{\sigma} \sim \text{Normal}\,(0,1)$$

1. $P(X \le a) = P\left(Z \le \frac{a-\mu}{\sigma}\right) = \text{From the table.}$
2. $P(X \ge a) = 1 - P(X \le a) = 1 - P\left(Z \le \frac{a-\mu}{\sigma}\right)$
3. $P(a \le X \le b) = P(X \le b) - P(X \le a)$

   $= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$
4. $P(X = a) = 0$, for every a.

- **Normal Distribution Application:**

Example:

Suppose that the hemoglobin levels of healthy adult males are approximately normally distributed with a mean of 16 and a variance of 0.81.

(a) Find that probability that a randomly chosen healthy adult male has a hemoglobin level less than 14.

(b) What is the percentage of healthy adult males who have hemoglobin level less than 14?

(c) In a population of 10,000 healthy adult males, how many would you expect to have hemoglobin level less than 14?

Solution:

X = hemoglobin level for healthy adults males

Mean: $\mu = 16$

Variance: $\sigma^2 = 0.81$

Standard deviation: $\sigma = 0.9$

$X \sim \text{Normal}(16, 0.81)$

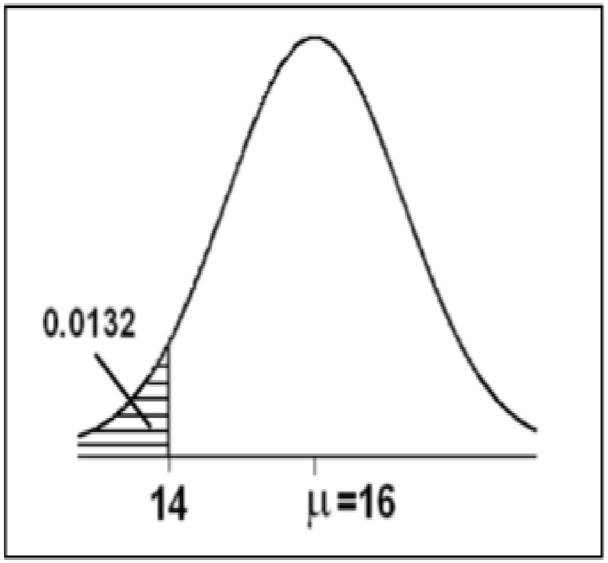
(a) The probability that a randomly chosen healthy adult male has hemoglobin level less than 14 is $P(X \leq 14)$.

$$P(X \leq 14) = P\left(Z \leq \frac{14-\mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{14-16}{0.9}\right)$$

$$= P(Z \leq -2.22)$$

$$= 0.0132$$



(b) The percentage of healthy adult males who have hemoglobin level less than 14 is:

$$P(X \le 14) \times 100\% = 0.0132 \times 100\% = 1.32\%$$

(c) In a population of 10000 healthy adult males, we would expect that the number of males with hemoglobin level less than 14 to be:

$$P(X \le 14) \times 10000 = 0.0132 \times 10000 = 132 \textit{ males}$$

Example:

Suppose that the birth weight of Saudi babies has a normal distribution with mean μ=3.4 and standard deviation σ=0.35.

(a) Find the probability that a randomly chosen Saudi baby has a birth weight between 3.0 and 4.0 kg.

(b) What is the percentage of Saudi babies who have a birth weight between 3.0 and 4.0 kg?

(c) In a population of 100000 Saudi babies, how many would you expect to have birth weight between 3.0 and 4.0 kg?

Solution:

X = birth weight of Saudi babies

Mean: $\mu = 3.4$

Standard deviation: $\sigma = 0.35$

Variance: $\sigma^2 = (0.35)^2 = 0.1225$

$X \sim \text{Normal}(3.4, 0.1225)$

(a) The probability that a randomly chosen Saudi baby has a birth weight between 3.0 and 4.0 kg is $P(3.0 < X < 4.0)$

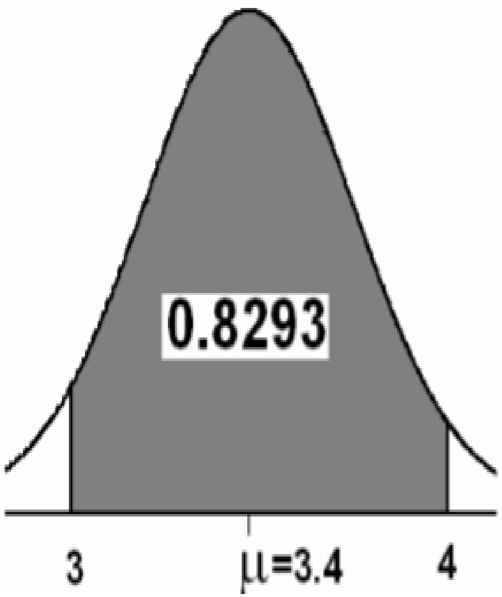
$$P(3.0 < X < 4.0) = P(X \le 4.0) - P(X \le 3.0)$$

$$= P\left(Z \le \frac{4.0 - \mu}{\sigma}\right) - P\left(Z \le \frac{3.0 - \mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{4.0 - 3.4}{0.35}\right) - P\left(Z \le \frac{3.0 - 3.4}{0.35}\right)$$

$$= P(Z \le 1.71) - P(Z \le -1.14)$$

$$= 0.9564 - 0.1271 = 0.8293$$



(b) The percentage of Saudi babies who have a birth weight between 3.0 and 4.0 kg is

$$P(3.0<X<4.0) \times 100\% = 0.8293 \times 100\% = 82.93\%$$

(c) In a population of 100,000 Saudi babies, we would expect that the number of babies with birth weight between 3.0 and 4.0 kg to be:

$$P(3.0<X<4.0) \times 100000 = 0.8293 \times 100000 = 82930 \text{ babies}$$

Application of the Normal Distribution

Example

In an industrial process, the diameter of a ball bearing is an important component part. The buyer sets specifications on the diameter to be 3.00 ± 0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that, in the process, the diameter of a ball bearing has a normal distribution with mean 3.00 cm and standard deviation 0.005 cm. On the average, how many manufactured ball bearings will be scrapped?

Solution:

$\mu=3.00$

$\sigma=0.005$

X=diameter

$X\sim N(3.00,\ 0.005)$

The specification limits are:

3.00 ± 0.01

x_1=Lower limit=3.00–0.01=2.99

x_2=Upper limit=3.00+0.01=3.01

$$P(x_1 < X < x_2) = P(2.99 < X < 3.01) = P(X < 3.01) - P(X < 2.99)$$

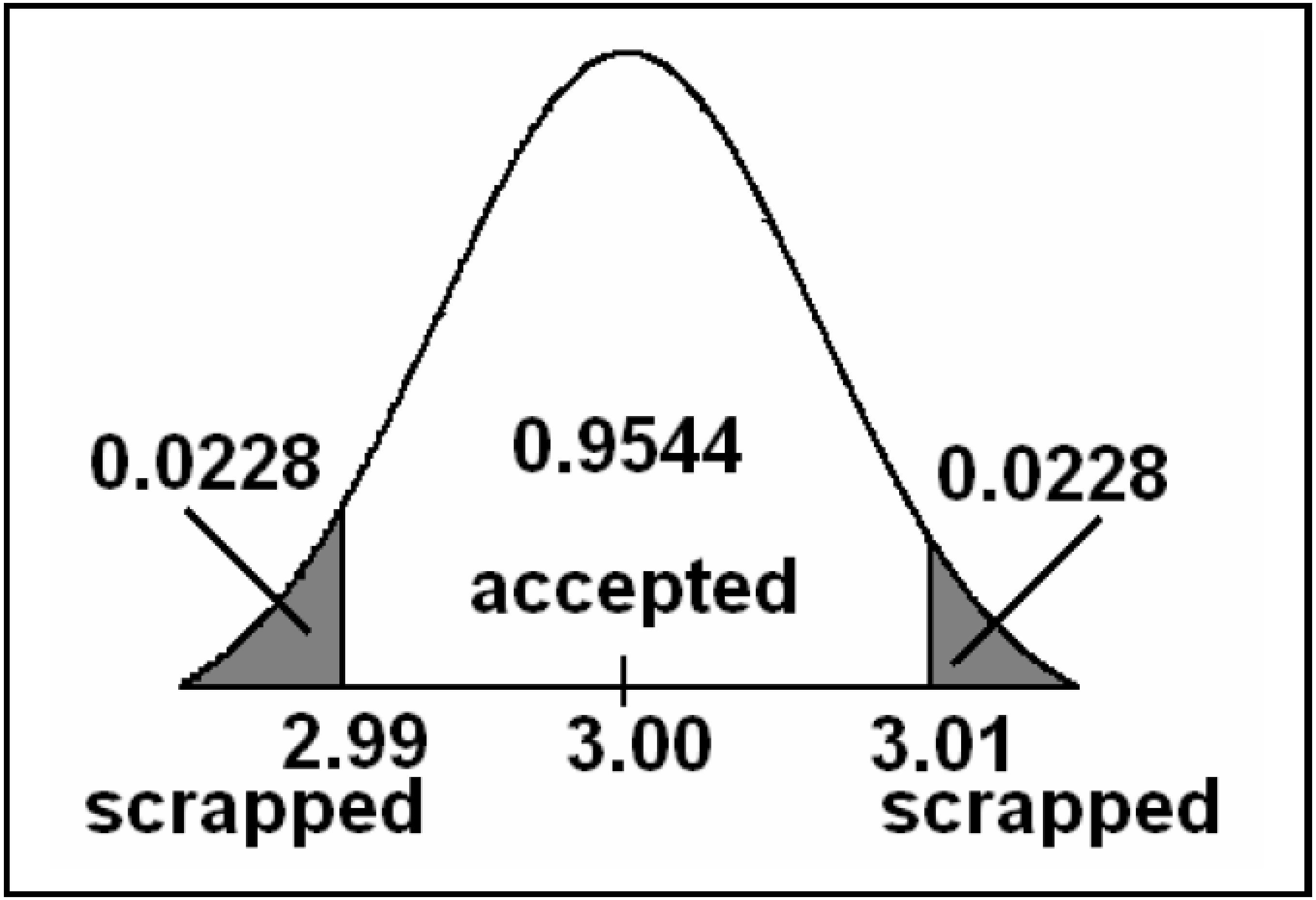
$$= P\left(Z \le \frac{3.01 - \mu}{\sigma}\right) - P\left(Z \le \frac{2.99 - \mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{3.01 - 3.00}{0.005}\right) - P\left(Z \le \frac{2.99 - 3.00}{0.005}\right)$$

$$= P(Z \le 2.00) - P(Z \le -2.00)$$

$$= 0.9772 - 0.0228$$

$$= 0.9544$$



Therefore, on the average, 95.44% of manufactured ball bearings will be accepted and 4.56% will be scrapped.

Example

Gauges are use to reject all components where a certain dimension is not within the specifications $1.50 \pm d$. It is known that this measurement is normally distributed with mean 1.50 and standard deviation 0.20. Determine the value d such that the specifications cover 95% of the measurements.

Solution:

$\mu=1.5$

$\sigma=0.20$

X= measurement

$X \sim N(1.5, 0.20)$

The specification limits are

$1.5 \pm d$

x_1=Lower limit=$1.5-d$

x_2=Upper limit=$1.5+d$

$$P(X > 1.5 + d) = 0.025 \Leftrightarrow P(X < 1.5 + d) = 0.975$$

$$P(X < 1.5 - d) = 0.025$$

$$P(X < 1.5 - d) = 0.025$$

$$\Leftrightarrow P\left(\frac{X-\mu}{\sigma} \le \frac{(1.5-d)-\mu}{\sigma}\right) = 0.025$$

$$\Leftrightarrow P\left(Z \leq \frac{(1.5-d)-\mu}{\sigma}\right) = 0.025$$

$$\Leftrightarrow P\left(Z \leq \frac{(1.5-d)-1.5}{0.20}\right) = 0.025$$

$$\Leftrightarrow P\left(Z \leq \frac{-d}{0.20}\right) = 0.025$$

$$\Leftrightarrow \frac{-d}{0.20} = -1.96$$

$$\Leftrightarrow -d = (0.20)(-1.96)$$

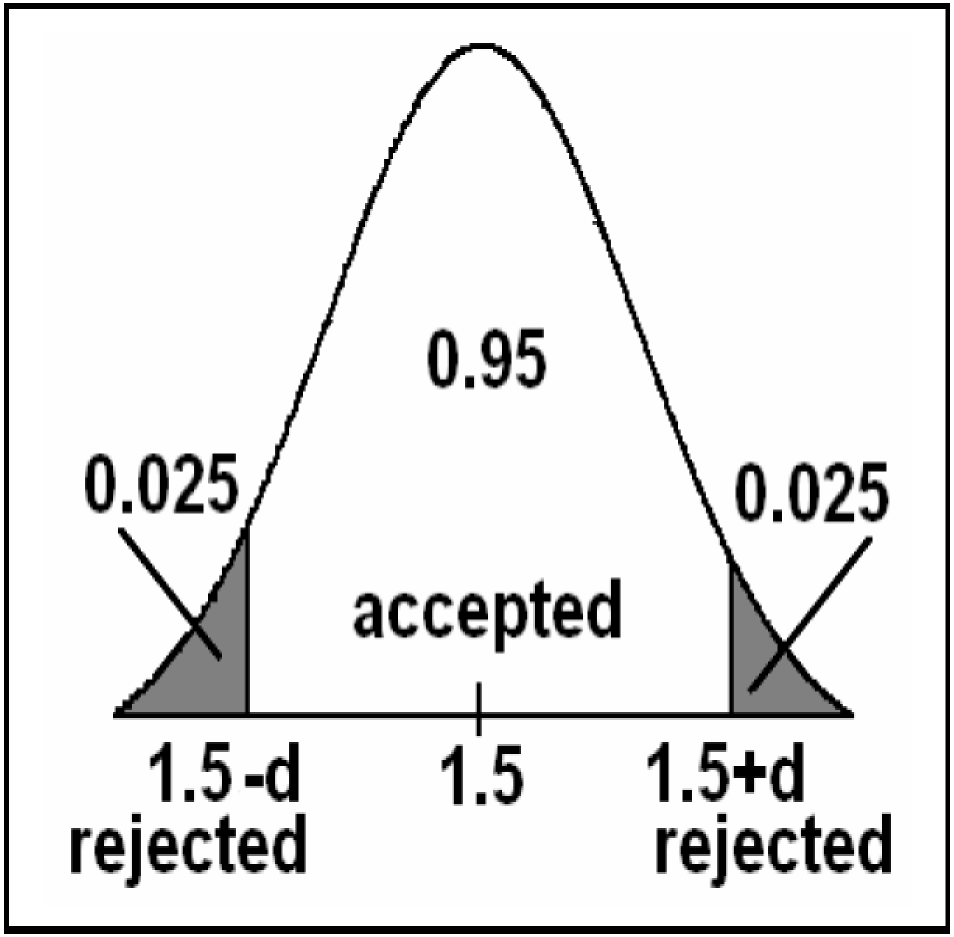
$$\Leftrightarrow d = 0.392$$

Z	…	0.06
⋮	⋮	⇑ ⇑
-1.9	⇐⇐	0.025

$$P\left(Z \le \frac{-d}{0.20}\right) = 0.025$$

$$\frac{-d}{0.20} = -1.96$$

$$\text{Note: } \frac{-d}{0.20} = Z_{0.025}$$



The specification limits are:

x_1=Lower limit=1.5−d = 1.5 − 0.392 = 1.108

x_2=Upper limit=1.5+d=1.5+0.392= 1.892

Therefore, 95% of the measurements fall within the specifications (1.108, 1.892).

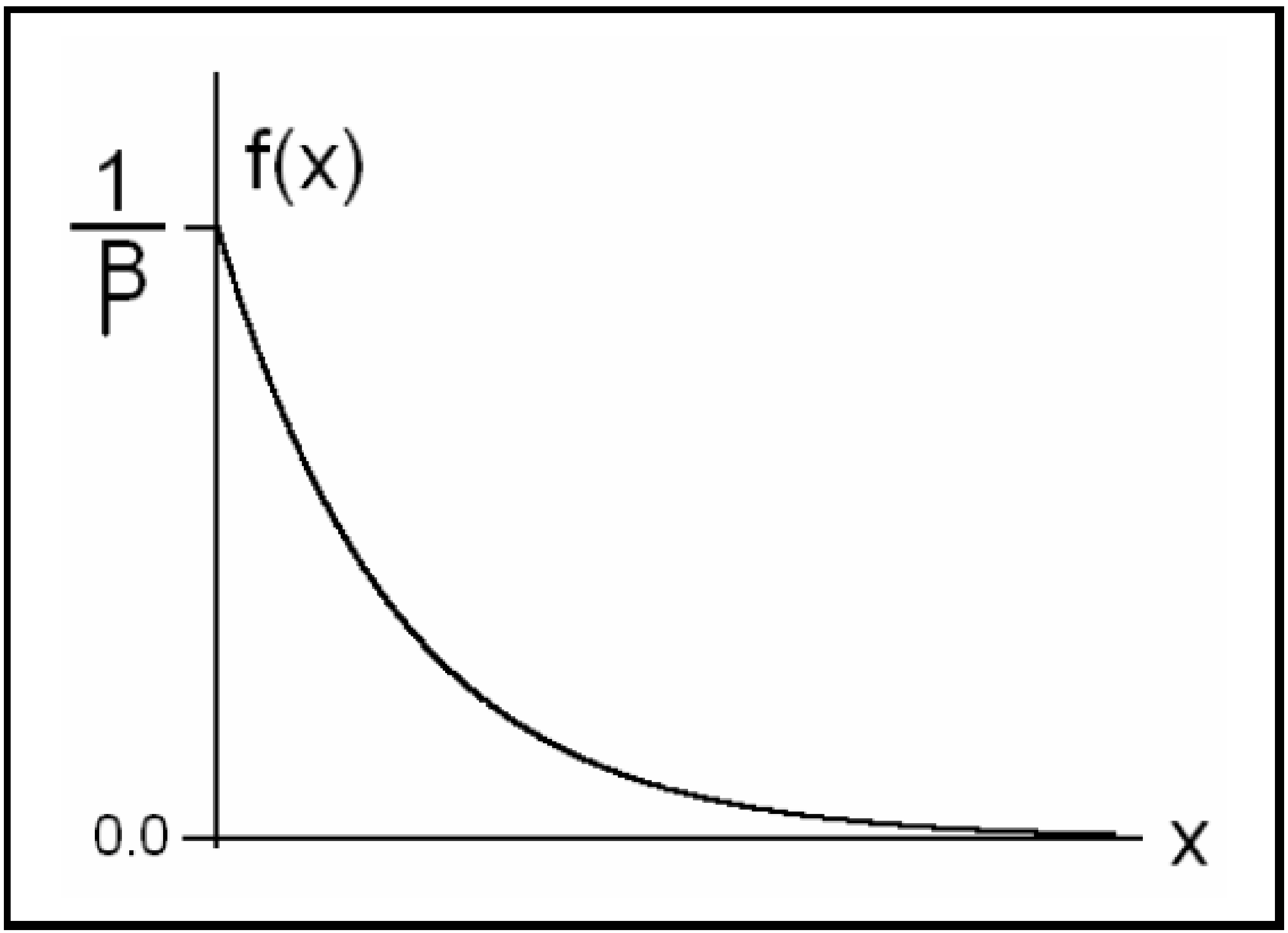
Exponential Distribution

Definition:

The continuous random variable X has an exponential distribution with parameter β, if its probability density function is given by:

$$f(x)=\begin{cases}\frac{1}{\beta}\, e^{-x/\beta} & ; x>0 \\ 0 \;;\; elsewhere & \end{cases}$$

and we write $X \sim Exp(\beta)$


f(x)
1
β
0.0
X

Theorem:

If the random variable X has an exponential distribution with parameter β, i.e., $X \sim Exp(\beta)$, then the mean and the variance of X are:

$$E(X) = \mu = \beta$$

$$Var(X) = \sigma^2 = \beta^2$$

Example

Suppose that a system contains a certain type of component whose time in years to failure is given by *T*. The random variable *T* is modeled nicely by the exponential distribution with mean time to failure $\beta=5$. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Solution:

$\beta = 5, \qquad T \sim Exp(5)$

The pdf of T is:

$$f(t) = \begin{cases} \frac{1}{5} e^{-t/5} & ; t > 0 \\ 0 & ; elsewhere \end{cases}$$

The probability that a given component is still functioning after 8 years is given by:

$$P(T > 8) = \int_8^\infty f(t)\, dt = \int_8^\infty \frac{1}{5} e^{-t/5}\, dt = e^{-8/5} = 0.2$$

Now define the random variable:

X= number of components functioning

after 8 years out of 5 components

$X \sim \text{Binomial}(5,\ 0.2) \qquad (n=5,\ p=P(T>8)=0.2)$

$$f(x) = P(X=x) = b(x;5,0.2) = \begin{cases} \binom{5}{x} 0.2^{x}\ 0.8^{5-x}\ ; & x = 0, 1, \ldots, 5 \\ 0\ ; & otherwise \end{cases}$$

The probability that at least 2 are still functioning at the end of 8 years is:

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)]$$

$$= 1 - [\binom{5}{0} 0.2^0 \, 0.8^{5-0} + \binom{5}{1} 0.2^1 \, 0.8^{5-1}]$$

$$= 1 - [\, 0.8^5 + 5 \times 0.2 \;\; \times 0.8^4]$$

$$= 1 - 0.7373$$

$$= 0.2627$$