CSC311 – Spring 2017
Design and Analysis of Algorithms
1. Introduction

(Chap. 1 – Introduction to Algorithms (3rd edition)
by Cormen, Leiserson, Rivest & Stein)

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Outline

- Mathematical essentials
- Algorithms
- Correct algorithms
- Data structures
- Technique
- Hard problems
- Choosing algorithms
- Design of algorithms
- Analysis of algorithms

2

Mathematical essentials

- Monotonicity

A function $f(n)$ is *monotonically increasing* if:

$$m \leq n \Rightarrow f(m) \leq f(n)$$

A function $f(n)$ is *monotonically decreasing* if:

$$m \leq n \Rightarrow f(m) \geq f(n)$$

A function $f(n)$ is *strictly increasing* if:

$$m < n \Rightarrow f(m) < f(n)$$

A function $f(n)$ is *strictly decreasing* if

$$m < n \Rightarrow f(m) > f(n)$$

3

Mathematical essentials

- Floors and ceilings

For any real number x, the greatest integer less than or equal to x is denoted by $\lfloor x \rfloor$.

For any real number x, the least integer greater than or equal to x is denoted by $\lceil x \rceil$.

For all real numbers x,

$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$.

Both functions are *monotonically increasing*.

4

Mathematical essentials

- Exponentials

  For all n and $a \geq 1$, the function a^n is the exponential function with base a and is *monotonically increasing*.

- Logarithms

$\log n = \log_2 n$	(binary logarithm),
$\ln n = \log_e n$	(natural logarithm),
$\log^k n = (\log n)^k$	(exponentiation),
$\log \log n = \log(\log n)$	(composition),
$\log n + k = (\log n)+k$	(precedence of log).

5

Mathematical essentials

- Logarithms

  For all real $a > 0, b > 0, c > 0$, and n

$$
\begin{aligned}
a &= b^{\log_b a} \\
\log_c(ab) &= \log_c a + \log_c b \\
\log_b a^n &= n \log_b a \\
\log_b a &= \frac{\log_c a}{\log_c b} \\
\log_b(1/a) &= -\log_b a \\
\log_b a &= \frac{1}{\log_a b} \\
a^{\log_b c} &= c^{\log_b a}
\end{aligned}
$$

6

Mathematical essentials

- Factorials

For all n the function $n!$ or "n factorial" is given by
$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1$

7

Mathematical essentials

- Functional iteration

The notation $f^{(i)}(n)$ represents the function $f(n)$ iteratively applied i times to an initial value of n, or, recursively

$f^{(i)}(n) = n$ if $i=0$

$f^{(i)}(n) = f(f^{(i-1)}(n))$ if $i>0$

Example:

If $f(n) = 2n$

then $f^{(2)}(n) = f(2n) = 2(2n) = 2^2 n$

then $f^{(3)}(n) = f(f^{(2)}(n)) = 2(2^2 n) = 2^3 n$

then $f^{(i)}(n) = 2^i n$

8

Mathematical essentials

- Iterated logarithmic function

The notation $\log^* n$ which reads "log star of n" is defined as

$$\log^* n = \min\ \{i \geq 0 : \log^{(i)} n \leq 1\}$$

Example:

$\log^* 2 = 1$

$\log^* 4 = 2$

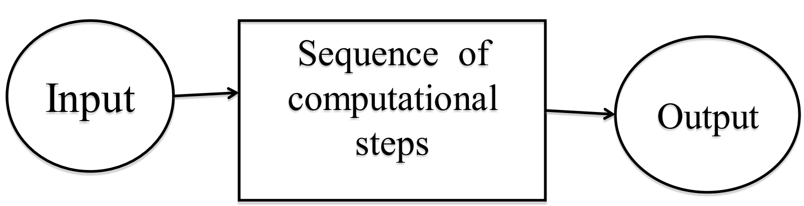
$\log^* 16 = 3$

$\log^* 65536 = 4$

$\log^* 2^{65536} = 5$

Algorithms

An algorithm is any well-defined computational procedure that takes some value, or set of values, as input and produces some value, or set of values, as output.



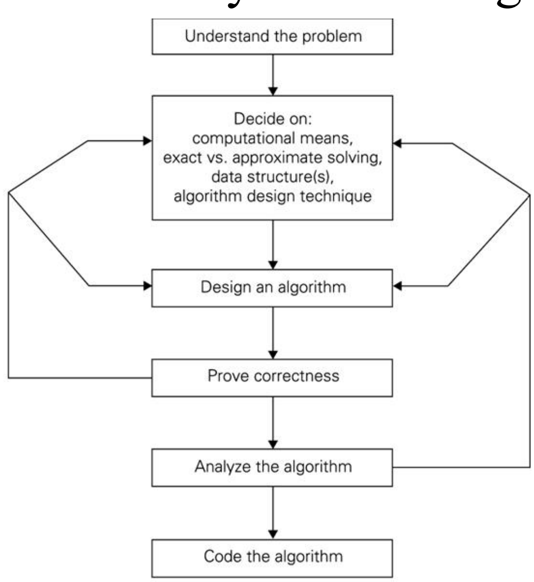
Example: Sorting problem

- Input: A sequence of n numbers: $a_1, a_2, \ldots, a_n$
- Output: A permutation (reordering) $a_i, a_j, \ldots, a_k$ of the input sequence such that $a_i \le a_j \le \ldots \le a_k$

- Ex. Input: sequence 31, 41, 59, 26, 41, 58
  Output: sequence 26, 31, 41, 41, 58, 59

11

Design and Analysis of an Algorithm



12

Correct Algorithms

- An algorithm is said to be correct if, for every input instance, it halts with the correct output. We say that a correct algorithm solves the given computational problem.
- An incorrect algorithm might not halt at all on some input instances, or it might halt with an answer other than the desired one.

13

What kinds of problems are solved by algorithms?

- We are given a road map on which the distance between each pair of adjacent intersections is marked, and our goal is to determine the shortest route from one intersection to another.
- We are given a sequence $A_1, A_2, \ldots, A_n$ of n matrices, and we wish to determine their product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$
- We are given an equation $ax \equiv b(\bmod n)$, where a, b, and n are integers, and we wish to find all the integers x, modulo n, that satisfy the equation.
- We are given n points in the plane, and we wish to find the convex hull of these points (the smallest convex polygon containing the points).

14

Data structures

- A data structure is a way to store and organize data in order to facilitate access and modifications.
- No single data structure works well for all purposes, and so it is important to know the strengths and limitations of several of them:
  - Table, Stacks and Queues, Linked lists
  - Representing rooted trees
  - Hash tables
  - Binary Search Trees
  - Red-black trees, …

15

Technique

- Techniques of algorithm design and analysis are used to develop algorithms, to show that they give the correct answer, and to understand their efficiency.

16

Hard problems

- There are some problems for which no efficient solution is known, which are known as NP-complete:
  - it is unknown whether or not efficient algorithms exist for NP-complete problems.

Pseudocode conventions

- **algorithm** indicates the beginning of the algorithm.
- Indentation indicates block structure.
- **while**, **for**, **repeat-until**, and **if-else** have same interpretations similar to Java.
  - for i=1 to n {

```
      x = x+i
      y = y+x
      …}
```

- // indicates the beginning of a comment
- Array A[1..n]; array elements: A[i], A[i+2] …
- **return** statement returns the control back to the point of call in the calling procedure.

Choosing algorithms

Ex: Fibonacci sequence is defined as follows.

$$F(0) = 0, F(1) = 1, \text{ and}$$
$$F(n) = F(n-1) + F(n-2) \text{ for } n > 1.$$

Write an algorithm to computer F(n).

There are many algorithms, but what is the most efficient one?

19

Algorithms 1 and 2 for Fibonacci

```
function fib1(n){
   if n < 2 then return n;
   else return fib1(n-1) + fib1(n-2);
}

function fib2(n){
   i= 1; j = 0;
   for k = 1 to n do { j = i+j; i = j- i;}
   return j;
}
```

20

Algorithm 3 for Fibonacci

```
function  fib3(n){
    i = 1; j = 0; k = 0; h = 1;
    while n>0 do {
                if (n odd) then {  t = jh;
                                j = ih + jk +t;
                                i = ik +t;}
                t = h^2;
                h = 2kh+t;
                k = k^2+t;
                n = n div 2;}
    return j;
}
```


Example of running times for Fibonacci

n	10	20	30	50	100	1000 0	1 000 000	1000 0000 0
fib1	8 ms	1 s	2 min	21 days				
fib2	1/6 ms	1/3 ms	½ ms	¾ ms	3/2 ms	150 ms	15 s	25 min
fib3	1/3 ms	2/5 ms	½ ms	½ ms	½ ms	1 ms	3/2 ms	2 ms

An Example: Insertion Sort

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

23

An Example: Insertion Sort

30	10	40	20
1	2	3	4

i = ∅ j = ∅ key = ∅
A[j] = ∅ A[j+1] = ∅

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

24

An Example: Insertion Sort

30	10	40	20
1	2	3	4

i = 2	j = 1	key = 10
A[j] = 30		A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

25

An Example: Insertion Sort

30	30	40	20
1	2	3	4

i = 2	j = 1	key = 10
A[j] = 30		A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

26

An Example: Insertion Sort

30	30	40	20
1	2	3	4

i = 2   j = 1   key = 10
A[j] = 30   A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
      }
      A[j+1] = key
  }
}
```


An Example: Insertion Sort

30	30	40	20
1	2	3	4

i = 2   j = 0   key = 10
A[j] = ∅   A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
      }
      A[j+1] = key
  }
}
```

An Example: Insertion Sort

30	30	40	20
1	2	3	4

i = 2 j = 0 key = 10
A[j] = ∅ A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

29

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 2 j = 0 key = 10
A[j] = ∅ A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

30

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 3 j = 0 key = 10
A[j] = ∅ A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

31

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 3 j = 0 key = 40
A[j] = ∅ A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

32

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 3 j = 0 key = 40
A[j] = ∅ A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
             A[j+1] = A[j]
             j = j - 1
      }
      A[j+1] = key
  }
}
```

33

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 3 j = 2 key = 40
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
             A[j+1] = A[j]
             j = j - 1
      }
      A[j+1] = key
  }
}
```

34

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 3 j = 2 key = 40
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

35

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 3 j = 2 key = 40
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

36

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 4 j = 2 key = 40
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
     key = A[i]
     j = i - 1;
     while (j > 0) and (A[j] > key) {
          A[j+1] = A[j]
          j = j - 1
     }
     A[j+1] = key
  }
}
```

37

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 4 j = 2 key = 20
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
     key = A[i]
     j = i - 1;
     while (j > 0) and (A[j] > key) {
          A[j+1] = A[j]
          j = j - 1
     }
     A[j+1] = key
  }
}
```

38

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 4 j = 2 key = 20
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
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            j = j - 1
      }
      A[j+1] = key
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}
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An Example: Insertion Sort

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1	2	3	4

i = 4 j = 3 key = 20
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InsertionSort(A, n) {
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An Example: Insertion Sort

10	30	40	40
1	2	3	4

i = 4 j = 3 key = 20
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InsertionSort(A, n) {
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An Example: Insertion Sort

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i = 4 j = 3 key = 20
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InsertionSort(A, n) {
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  }
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43

An Example: Insertion Sort

10	30	40	40
1	2	3	4

i = 4 j = 3 key = 20
A[j] = 40 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
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             j = j - 1
      }
      A[j+1] = key
  }
}
```

44

An Example: Insertion Sort

10	30	40	40
1	2	3	4

i = 4 j = 2 key = 20
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
    key = A[i]
    j = i - 1;
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45

An Example: Insertion Sort

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  }
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```

46

An Example: Insertion Sort

10	30	30	40
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i = 4 j = 2 key = 20
A[j] = 30 A[j+1] = 30

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InsertionSort(A, n) {
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  }
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```

47

An Example: Insertion Sort

10	30	30	40
1	2	3	4

i = 4 j = 2 key = 20
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```
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  for i = 2 to n {
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      j = i - 1;
      while (j > 0) and (A[j] > key) {
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             j = j - 1
      }
      A[j+1] = key
  }
}
```

48

An Example: Insertion Sort

10	30	30	40
1	2	3	4

i = 4 j = 1 key = 20
A[j] = 10 A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
     key = A[i]
     j = i - 1;
     while (j > 0) and (A[j] > key) {
          A[j+1] = A[j]
          j = j - 1
     }
     A[j+1] = key
  }
}
```

49

An Example: Insertion Sort

10	30	30	40
1	2	3	4

i = 4 j = 1 key = 20
A[j] = 10 A[j+1] = 30

```
InsertionSort(A, n) {
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          j = j - 1
     }
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  }
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```

50

An Example: Insertion Sort

10	20	30	40
1	2	3	4

i = 4 j = 1 key = 20
A[j] = 10 A[j+1] = 20

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
             A[j+1] = A[j]
             j = j - 1
      }
      A[j+1] = key
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An Example: Insertion Sort

10	20	30	40
1	2	3	4

i = 4 j = 1 key = 20
A[j] = 10 A[j+1] = 20

```
InsertionSort(A, n) {
  for i = 2 to n {
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      while (j > 0) and (A[j] > key) {
             A[j+1] = A[j]
             j = j - 1
      }
      A[j+1] = key
  }
}
```

Done!

Design of algorithms

Example:

- The divide-and-conquer approach
  - Divide the problem into a number of subproblems.
  - Conquer the subproblems by solving them recursively. If the subproblem sizes are small enough, however, just solve the subproblems in a straightforward manner.
  - Combine the solutions to the subproblems into the solution for the original problem.
- Recursive structure: to solve a given problem, they call themselves recursively one or more times to deal with closely related subproblems.

53

Analysis of Algorithms

- Predicting the required resources
- What do we measure?
  - Computational time
  - Memory
  - Communication bandwidth
  - Other

Analysis of Algorithms

- Analysis is performed with respect to a computational model
- We will usually use a generic uniprocessor random-access machine (RAM)
  - All memory equally expensive to access
  - No concurrent operations
  - All reasonable instructions take unit time
    - Except, of course, function calls
  - Constant word size
    - Unless we are explicitly manipulating bits

55

Input Size

- Time and space complexities
  - This is generally a function of the input size
    - e.g., sorting, multiplication
  - How we characterize input size depends:
    - Sorting: number of input items
    - Multiplication: total number of bits
    - Graph algorithms: number of nodes & edges
    - etc.

56

Running Time

- Number of primitive steps that are executed
  - Except for time of executing a function call, most statements roughly require the same amount of time
    - $y = m * x + b$
    - $c = 5 / 9 * (t - 32)$
    - $z = f(x) + g(y)$
- We can be more exact if needed

57

Worst-case and average-case analysis

- worst-case running time: the longest running time for any input of size n:
  - upper bound on the running time for any input
  - for some algorithms, the worst case occurs fairly often
  - the "average case" is often roughly as bad as the worst case.
- average-case or expected running time:
  - technique of probabilistic analysis
  - assume that all inputs of a given size are equally likely
  - difficult to analyze.

58

Reading

Chapter 1

Cormen, Leiserson, Rivest, & Stein, Introduction to Algorithms (Third Edition), The MIT Press, 2009.

59