THE MIXED BOUNDARY VALUE PROBLEM FOR HELMHOLTZ' EQUATION IN A SECTOR: SOME EXISTENCE RESULTS

M.A. Al-Gwaiz
College of Engineering, University of Riyadh, Saudi Arabia

Abstract. Under certain assumptions, the motion of water waves over a uniformly sloping beach may be posed as a mixed boundary value problem for Hemholtz' equation in a semi infinite sector. A class of solutions to this problem was constructed by Ursell [14] and Roseaux [12]. Here we shed some light on the completeness of these solutions.

1. Formulation of the problem

We consider an inviscid, incompressible and irrotational fluid of constant density. At rest it occupies the region

$$\{(r,\theta,z) : 0 \leq r < \infty, -\gamma \leq \theta < 0, -\infty < z < \infty\},$$

where γ is the beach angle satisfying $0 < \gamma < \pi$. The $z-$axis coincides with the shoreline and, with $\theta = 0$, the $r-$axis lies on the undisturbed surface of the fluid. This region will be denoted by the cartesian product $(0,\infty) \times [-\gamma, 0] \times (-\infty, \infty)$ in cylindrical coordinates.

The velocity potential function $\Phi(r,\theta,z,t) = e^{i(\omega t + kz)}\phi(r,\theta)$, where t is the time variable and ω and k are real constants, represents a time-harmonic wave which is periodic in the z direction. Such waves may appear on the ocean surface as progressing waves whose wavefronts form parallel lines at an arbitrary angle with the shoreline. In the absence of external forces, the system of linearized equations which $\phi(r,\theta)$ must satisfy (see [13]) for example) is given by

$$\left\{ \begin{array}{l} \Delta\phi = k^2\phi \quad \text{in } (0,\infty) \times (-\gamma, 0), \\ \frac{1}{r}\phi_\theta = \frac{\omega^2}{g}\phi \quad \text{on } r > 0, \theta = 0, \\ \phi_\theta = 0 \quad \text{on } r > 0, \theta = -\gamma, \end{array} \right. \tag{1.1}$$

where g is the gravitational acceleration constant and Δ is the Laplacian operator in two dimensions. These equations also describe the motion of a fluid confined to the canal $(0,\infty)\times[-\gamma,0]\times[0,\pi/k]$ if the velocity potential is given by $(e^{i\omega t}\cos kz)\,\phi(r,\theta)$ [14].

The case when $k=0$ yields the two dimensional problem for Laplace's equation, which has been completely solved by Peters and Roseau (see [9] and [12]). We therefore confine ourselves to the case when $k>0$. If the positive constant k is absorbed into r by a change of variable, Equations (1.1) become

$$\left\{\begin{array}{ll}\Delta\phi=\phi & \text{in } (0,\infty)\times(-\gamma,0)\,,\\ \frac{1}{r}\phi_\theta=\lambda\phi & \text{on } r>0,\theta=0,\\ \phi_\theta=0 & \text{on } r>0,\theta=-\gamma,\end{array}\right. \tag{1.2}$$

where $\lambda=\omega^2/gk$ is a positive parameter which depends on the frequency ω when k is constant.

At the outset we assume that ϕ is twice differentiable in the interior of the sector $(0,\infty)\times(-\gamma,0)$ and has continuous first derivatives up to the boundary $r>0,\ \theta=0,-\gamma$. Physical considerations dictate that ϕ should be bounded at $r=\infty$ and at most logarithmically singular at $r=0$. The singularity at $r=0$ allows waves to break along the shoreline, with the resulting energy dissipation, and it has the appropriate strength for an energy sink in two dimensions. We thus restrict ϕ to the class of functions $u\in C^2((0,\infty)\times(-\gamma,0))\cap C^1((0,\infty)\times[-\gamma,0])$ such that

$$u(r,\theta)=\left\{\begin{array}{ll}O(\log r) & \text{as } r\to 0\\ O(1) & \text{as } r\to\infty,\end{array}\right.$$

for all $\theta\in[-\gamma,0]$. We shall denote such a class by W.

We distinguish between the solutions of (1.2) which are bounded at $r=0$, representing reflected waves, and those which are singular. As it turns out, the frequency parameter λ has a significant effect on the type of solution obtained. An investigation of this relationship between the value of λ and the behaviour of the solution to (1.2) near $r=0$, in particular for $0<\lambda<1$, forms the subject of this paper.

The boundary value problem (1.2) has been solved by Peters [9] and Roseau [12] for the case when $\lambda>1$ and $0<\gamma<\pi$. There are two unique and linearly independent solutions in W for each value of $\lambda\in(1,\infty)$: one is bounded and the other singular at the origin. Unless otherwise stated, we shall assume that $0<\lambda<1$ in all that follows.

For $0 < \gamma < \pi/2$ Ursell [14] constructed a bounded solution to (1.2) for each $\lambda = \sin(2\nu - 1)\gamma$, where $\nu \in \{1, 2, 3, \cdots\}$ and $(2\nu - 1)\gamma \leq \pi/2$. The first solution, corresponding to $\nu = 1$, is the so-called Stokes wave $e^{-r\cos(\theta+\gamma)}$ which is always present when $0 < \gamma < \pi/2$. Since the number of solutions of the inequality $(2\nu - 1)\gamma \leq \pi/2$ is $\left[\frac{\pi}{4\gamma} + \frac{1}{2}\right]$, the greatest integer contained in $\frac{\pi}{4\gamma} + \frac{1}{2}$, there exist at least as many bounded solutions to (1.2) for a given γ. When $\lambda \neq \sin(2\nu - 1)\gamma$ and provided $\gamma \neq \pi/2m$, where m is a positive integer, Roseau [12] found that a singular solution to (1.2) can always be constructed.

In this paper we shall establish the following results:
(i) When $0 < \gamma < \pi/2$, there is at least one solution (the Stokes wave) and at most a finite number of bounded solutions. When $\pi/2 \leq \gamma < \pi$, the boundary value problem (1.2) has no bounded solutions.
(ii) In the absence of a bounded solution, a singular solution always appears, and the two cannot exist together, i.e. for the same value of λ. Furthermore, both solutions are unique (up to a multiplicative constant).

Thus the restriction $\gamma \neq \pi/2m$ mentioned above appears to be a limitation of the method used by Roseau, and may be dropped. The assertions in (ii) are confirmed by Lehman and Lewy [6] through a different method. The approach adopted here is based on the spectral theory of compact operators on Hilbert space and the Fredholm alternative theorem for integral equations. Using some estimates on ϕ near $r = 0$ and $r = \infty$, which were obtained in [6], we first reduce the boundary value problem (1.2) to an integral equation through Green's theorem. Then the integral equation is posed as an eigenvalue problem for a bounded, self-adjoint operator in $L^2(0, \infty)$. The extended Fredholm theory for compact operators, in addition to some norm estimates, finally lead to the conclusions (i) and (ii).

2. The asymptotic behaviour of ϕ

For the purpose of applying Green's identity to the sector $(0, \infty) \times (-\gamma, 0)$ we need to have some estimates on the solution of (1.2) and its first derivatives in the neighbourhoods of $r = 0$ and $r = \infty$. The following lemma gives a bound on $|\nabla u(r, \theta)|$ in terms of the maximum value of $|u|$ over a disc centered at (r, θ), where u is a solution of the Helmholtz equation.

Lemma 1 *Let $u(r,\theta)$ be a solution of $\Delta u = u$ in an open connected set Ω in the plane. Let the disc $D(r,\theta,\rho) = \left\{(r,\theta) : \left|r'e^{i\theta'} - re^{i\theta}\right| \le \rho\right\}$, where $\rho > 0$, lie in Ω. Then*

$$|\nabla u(r,\theta)| \le \frac{3}{2\rho} \max_{(r',\theta')\in D(r,\theta,\rho)} |u(r',\theta')|$$

Proof. The function $v(r,\theta,z) = e^{iz}u(r,\theta)$ is harmonic in the open set $\Omega \times (-\infty,\infty)$. From the known properties of harmonic functions (see [10], chapter 2, section 13), we can write

$$\begin{aligned} |\nabla v(r,\theta,0)| &\le \frac{3}{2\rho} \max_{|(r',\theta',z')-(r,\theta,0)|=\rho} |v(r',\theta',z')| \\ &= \frac{3}{2\rho} \max_{(r',\theta')\in D(r,\theta,\rho)} |u(r',\theta')| . \end{aligned}$$

Since $|\nabla u(r,\theta)| \le |\nabla v(r,\theta,0)|$, the lemma is proved. □

The behaviour of the solution of (1.2) near $r = 0$ has been investigated by Lewy [7] and Lehman [5]. Based on their results and Lemma 1, the following estimates on ϕ and ϕ_r may be obtained [6]:

Lemma 2 *Any solution of (1.2) in W has the following asymptotic behaviour as $r \to 0$ in $-\gamma \le \theta \le 0$:*

$$\begin{aligned} \phi(r,\theta) &= B\log r + A + o(1) \\ \phi_r(r,\theta) &= B\frac{1}{r} + o\left(\frac{1}{r}\right), \end{aligned}$$

where A and B are constants which do not both vanish.

It is worth noting that Lemma 2 already implies the uniqueness of the bounded solution, for if $B = 0$ then $A \ne 0$ and the bounded solution does not vanish at the origin. But the linear combination $\phi(r,\theta) - \frac{\phi(0,0)}{\phi^*(0,0)}\phi^*(r,\theta)$ of any two bounded solutions ϕ and ϕ^* is a bounded solution which vanishes at $r = 0$. By Lemma 2 it must vanish identically.

Lemma 3 *If ϕ is a solution of (1.2) in W, then $|\nabla u| = O(1)$ as $r \to \infty$ in $-\gamma \le \theta \le 0$.*

Proof. ϕ is clearly a bounded solution of $\Delta\phi = \phi$ in $(1,\infty)\times[-\gamma,0]$. Since $\phi_\theta(r,-\gamma)=0$, ϕ may be extended by the symmetry relation $\phi(r,\theta) = \phi(r,-\theta-2\gamma)$ into $(1,\infty)\times[-2\gamma,0]$, where it remains a bounded solution of $\Delta\phi=\phi$. We define

$$\begin{aligned}
U(r,\theta) &= \frac{\cos\theta}{r}\phi_\theta(r,\theta) + \sin\theta\phi_r(r,\theta) - \lambda\phi(r,\theta),\\
\tilde{\phi}(x,y) &= \phi(r,\theta),\\
\tilde{U}(x,y) &= U(r,\theta),
\end{aligned}$$

Where $x = r\cos\theta$, $y = r\sin\theta$, and the cartesian coordinates (x,y) are restricted to the sector $(0,\infty)\times(-\pi,\pi)$ in order to allow for the possibility that the extension of ϕ beyond $-\gamma\le\theta\le 0$ may not be single valued. Thus $\tilde{U}(x,y) = \tilde{\phi}_y(x,y) - \lambda\tilde{\phi}(x,y)$, which, by integration, yields

$$\tilde{\phi}(x,y) = e^{\lambda y}\tilde{\phi}(x,0) + \int_0^y e^{\lambda(y-\eta)}\tilde{U}(x,\eta)\,d\eta \tag{2.1}$$

on $x>1$, $-x|\tan\gamma|\le y\le 0$. Since $U(r,\theta)=0$ when $\theta=0$, this function may be extended from $(1,\infty)\times[-2\gamma,0]$ into $(1,\infty)\times[-2\gamma,2\gamma]$ by the antisymmetry relation $U(r,\theta) = -U(r,-\theta)$. Using this relation in (2.1) and integrating by parts,

$$\begin{aligned}
\tilde{\phi}(x,y) &= e^{\lambda y}\tilde{\phi}(x,0) - \int_0^y e^{\lambda(y-\eta)}\tilde{U}(x,-\eta)\,d\eta\\
&= e^{\lambda y}\tilde{\phi}(x,0) - \int_0^y e^{\lambda(y-\eta)}\tilde{\phi}_y(x,-\eta)\,d\eta + \lambda\int_0^y e^{\lambda(y-\eta)}\tilde{\phi}(x,-\eta)\,d\eta\\
&= \tilde{\phi}(x,-y) + 2\lambda\int_0^y e^{\lambda(y-\eta)}\tilde{\phi}(x,-\eta)\,d\eta.
\end{aligned}$$

Since $\tilde{\phi}$ is bounded in $x>1$, $-|\tan\gamma|\le y\le 0$, the above equation implies that it is also bounded in $x>1$, $0\le y\le|\tan\gamma|$. Thus ϕ is bounded in the sector $\{(r,\theta): r>1, -2\gamma\le\theta\le 0\}$ and in the rectangle $\{(x,y): x>1, 0\le y\le|\tan\gamma|\}$.

Now the application of lemma 1 with $\rho = \min\{|\tan\gamma|, 2\sin\gamma, 1\}$ implies that $|\nabla\phi(r,\theta)|$ is bounded in $(2,\infty)\times[-\gamma,0]$. □

By a similar technique the following lemma may be proved (see [6]).

Lemma 4 *Let ϕ be a solution of (1.2) in W for $0<\lambda<1$. Then $\phi(r,\theta)=ae^{-\sqrt{1-\lambda^2}r}+O\left(e^{-br\sin\gamma}\right)$ as $r\to\infty$, where a and b are constants, with $b>0$ and independent of λ and γ.*

3. The integral equation

We seek a Green function $G(r',\theta',r,\theta)$ which satisfies

$$(\Delta'-1)\,G=\left(\frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2}{\partial\theta'^2}-1\right)G=-\delta\left(\left|r'e^{i\theta'}-re^{i\theta}\right|\right)$$

in $\{(0,\infty)\times(-\gamma,0)\}\times\{(0,\infty)\times(-\gamma,0)\}$ whose normal derivative $\frac{1}{r'}\frac{\partial G}{\partial\theta'}$ vanishes along the boundary $\theta'=0,-\gamma$. Here Δ' is the Laplacian operator in the coordinates (r',θ'), and δ is the Dirac measure [2] in the plane. Such a function has already been investigated at some length in [4]. It is represented by

$$G(r',\theta',r,\theta)=\begin{cases}\frac{1}{\gamma}\sum_{\nu=0}^{\infty}\varepsilon_\nu I_{\nu\mu}(r')\,K_{\nu\mu}(r)\cos\nu\mu\theta'\cos\nu\mu\theta, & r'<r\\ \frac{1}{\gamma}\sum_{\nu=0}^{\infty}\varepsilon_\nu K_{\nu\mu}(r')\,I_{\nu\mu}(r)\cos\nu\mu\theta'\cos\nu\mu\theta, & r'>r,\end{cases}\tag{3.1}$$

where $\mu=\pi/\gamma>1$, $\varepsilon_0=1$, and $\varepsilon_\nu=2$ for $\nu=1,2,3,\cdots$. $I_\alpha(r)$ and $K_\alpha(r)$ are the modified Bessel functions, which are positive in $C^\infty(0,\infty)$ and have the following behaviour for all $\alpha\geq 0$:
As $r\to 0$,

$$\begin{aligned}I_\alpha(r)&\backsim\frac{r^\alpha}{2^\alpha\Gamma(\alpha+1)},\\ K_\alpha(r)&\backsim\frac{2^{\alpha-1}\Gamma(\alpha)}{r^\alpha},\quad\alpha>0,\\ K_0(r)&\backsim\log\frac{2}{r}.\end{aligned}$$

As $r\to\infty$,

$$\begin{aligned}I_\alpha(r)&\backsim\frac{1}{\sqrt{2\pi r}}e^r,\\ K_\alpha(r)&\backsim\sqrt{\frac{\pi}{2r}}e^{-r}.\end{aligned}$$

Thus $G(0,\theta',r,\theta)=\frac{1}{\gamma}I_0(0)\,K_0(r)=\frac{1}{\gamma}K_0(r)$. From the identity (see [8])

$$I_\alpha(r')\,K_\alpha(r)=\int_0^\infty I_{2\alpha}\left(2\sqrt{r'r}\sinh\rho\right)e^{-(r'+r)\cosh\rho}d\rho,$$

and the fact that $I_\alpha(\rho)$ is a positive, decreasing function of α on $\alpha\geq 0$ for every fixed $\rho\in[0,\infty)$, we conclude that the product $I_\alpha(r')\,K_\alpha(r)$ is also a monotonically decreasing function of α on $\alpha\in[0,\infty)$. With $r'<r$ we thus arrive at the estimate

$$\begin{aligned}|G(r',\theta',r,\theta)| &\leq G(r',0,r,0)\\ &= \frac{1}{\gamma}\sum_{\nu=0}^{\infty}\varepsilon_\nu I_{\nu\mu}(r')\,K_{\nu\mu}(r)\\ &\leq \frac{1}{\gamma}\sum_{\nu=0}^{\infty}\varepsilon_\nu I_{\nu m}(r')\,K_{\nu m}(r),\end{aligned}$$

where $m=[\mu]=$ greatest integer less or equal to μ. Therefore, assuming $r>r'$,

$$G(r',0,r,0)\leq\frac{1}{\gamma}\sum_{\nu=0}^{\infty}\varepsilon_\nu I_\nu(r')\,K_\nu(r)=\frac{1}{\gamma}K_0(r-r')$$

by the addition formula for Bessel functions (see [8]). From (3.1) we have the symmetry relation $G(r,\theta,r',\theta')=G(r',\theta',r,\theta)$, so the iequality

$$|G(r',\theta',r,\theta)|\leq G(r',0,r,0)\leq\frac{1}{\gamma}K_0(|r'-r|) \tag{3.2}$$

holds whenever $r'\neq r$.

Let $\phi\in W$ be a solution of the boundary value problem (1.2). Applying Green's identity to the sector $S=(\epsilon,R)\times(-\gamma,0)$,

$$\iint_S(G\Delta'\phi-\phi\Delta'G)\,d\tau=\int_{\partial S}\left(G\frac{\partial\phi}{\partial n'}-\phi\frac{\partial G}{\partial n'}\right)d\sigma,$$

where $\frac{\partial G}{\partial n'}$ is the normal derivative of G on the boundary ∂S of S in the (r',θ')

coordinates. Using the properties of ϕ and G, we obtain

$$\begin{aligned}\phi(r,\theta) &= -\int_{-\gamma}^{0}\left[G(\epsilon,\theta',r,\theta)\,\phi_{r'}(\epsilon,\theta')-\phi(\epsilon,\theta')\,G_{r'}(\epsilon,\theta',r,\theta)\right]\epsilon d\theta' \\ &\quad +\lambda\int_{\epsilon}^{R}G(r',0,r,\theta)\,\phi(r',0)\,dr' \\ &\quad +\int_{-\gamma}^{0}\left[G(R,\theta',r,\theta)\,\phi_{r'}(R,\theta')-\phi(R,\theta')\,G_{r'}(R,\theta',r,\theta)\right]Rd\theta'.\end{aligned}$$

As $R\to\infty$ the last integral in this equation tends to 0, since ϕ and $\phi_{r'}$ remain bounded, by Lemma 3, whereas G and $G_{r'}$ decay exponentially. By Lemma 2, the first integral tends to $-\gamma BG(0,\theta',r,\theta)=BK_0(r)$ as $\epsilon\to 0$. We therefore arrive at the integral equation

$$\phi(r,\theta)=-BK_0(r)+\lambda\int_0^{\infty}G(r',0,r,\theta)\,\phi(r',0)\,dr', \tag{3.3}$$

where B is the constant coefficient of $\log r$ in the asymptotic expansion of ϕ about $r=0$. When ϕ is a bounded solution of (1.2), this constant vanishes and the resulting integral equation is homogeneous.

Setting $\theta=0$ in (3.3), and denoting $\phi(r,0)$ by $\psi(r)$ and $G(r',0,r,0)$ by $G_0(r',r)$, equation (3.3) becomes

$$\psi(r)=-BK_0(r)+\lambda\int_0^{\infty}G_0(r',r)\,\psi(r')\,dr', \tag{3.4}$$

which we now consider as an integral equation in its own right, defined over the set of functions

$$W_0=\left\{u\in C^1(0,\infty):u(r)=O(\log r)\ \text{ as } r\to 0, u(r)=O(1)\ \text{ as } r\to\infty\right\}.$$

Theorem 1 *Every solution of (3.4) in W_0 is the restriction to $\theta=0$ of a solution of (1.2) in W. Conversely, every solution of (1.2) in W is an extension into $(0,\infty)\times[-\gamma,0]$ of a solution of (3.4) in W_0. The correspondence between the two sets of solutions is one-to-one.*

Proof. We have already shown that if $\phi(r,\theta)\in W$ satisfies (1.2) then $\phi(r,0)$ satisfies (3.4). If $\phi(r,0)\equiv 0$ then $B=0$ and (3.3) implies $\phi(r,\theta)$ vanishes identically. Now let $\psi\in W_0$ be a solution of (3.4) and define

$$\phi(r,\theta)=-BK_0(r)+\lambda\int_0^{\infty}G(r',0,r,\theta)\,\psi(r')\,dr',\quad -\gamma\le\theta\le 0,$$

which clearly lies in W and coincides with $\psi(r)$ on $\theta = 0$. To show that ϕ satisfies (1.2) we need only check the boundary condition at $\theta = 0$. For $-\gamma \le \theta \le 0$, we have

$$\phi_\theta(r,\theta) = \lambda \int_0^\infty G_\theta(r',0,r,\theta)\,\psi(r')\,dr'.$$

Since $G(r',0,r,\theta)$ is symmetric about $\theta = 0$, $G_\theta(r',0,r,\theta) \to 0$ as $\theta \to 0$ outside every neighbourhood of $r' = r$. As $\theta \to 0$ in the neighbourhood of $r' = r$,

$$G_\theta(r',0,r,\theta) = -\frac{1}{\pi}\log\left(\left|r' - re^{i\theta}\right|\right) + O(1)$$

and

$$\begin{aligned}\frac{1}{r}G_\theta(r',0,r,\theta) &= -\frac{1}{\pi}\frac{1}{|r' - re^{i\theta}|}\frac{1}{r}\frac{\partial}{\partial\theta}\left|r' - re^{i\theta}\right| + O(1)\\ &= \frac{1}{\pi}\frac{r'\sin\theta}{(r' - r\cos\theta)^2 + (r'\sin\theta)^2} + O(1).\end{aligned}$$

This is a "delta-convergent sequence" in the terminology of [2], in the sense that

$$\frac{1}{\pi}\frac{r'\sin\theta}{(r' - r\cos\theta)^2 + (r'\sin\theta)^2} \to \delta(r' - r) \quad \text{as } \theta \to 0^-.$$

Hence

$$\frac{1}{\pi}\int_0^\infty \frac{r'\sin\theta}{(r' - r\cos\theta)^2 + (r'\sin\theta)^2}\phi(r',0)\,dr' \to \phi(r,0) \quad \text{as } \theta \to 0^-.$$

□

On the basis of this theorem we may now restrict our attention to the integral equation (3.4) for the purpose of investigating existence questions regarding our original boundary value problem (1.2).

4. The integral operator in Hilbert space

Let $L^2(0,\infty)$ be the Hilbert space of real, square-integrable functions on $(0,\infty)$ with inner product $\langle u,v\rangle = \int_0^\infty u(r)\,v(r)\,dr$. For any $u \in L^2(0,\infty) \cap C(0,\infty)$ we define the linear operator

$$Tu(r) = \int_0^\infty G_0(r',r)\,u(r')\,dr'. \tag{4.1}$$

Using the inequality (3.2), we have

$$
\begin{aligned}
\|Tu\|^2 &= \int_0^\infty [Tu(r)]^2\, dr \\
&= \int_0^\infty \left[\int_0^\infty G_0(r', r)\, u(r')\, dr'\right]^2 dr \\
&\le \frac{1}{\gamma^2}\int_0^\infty \left[\int_0^\infty K_0(|r'-r|)\, |u(r')|\, dr'\right]^2 dr \\
&= \frac{1}{\gamma^2}\int_0^\infty \left[\int_0^r K_0(r-r')\, |u(r')|\, dr' + \int_r^\infty K_0(r'-r)\, |u(r')|\, dr'\right]^2 dr \\
&= \frac{1}{\gamma^2}\int_0^\infty \left[\int_0^r K_0(\rho)\, |u(r-\rho)|\, d\rho + \int_0^\infty K_0(\rho)\, |u(r+\rho)|\, d\rho\right]^2 dr \\
&\le \frac{1}{\gamma^2}\int_0^\infty \left[\int_0^\infty K_0(\rho)\, \{|u(|r-\rho|)| + |u(r+\rho)|\}\, d\rho\right]^2 dr.
\end{aligned}
$$

The integrand $\left[\int_0^r K_0(\rho)\, |u(r-\rho)|\, d\rho + \int_r^\infty K_0(\rho)\, |u(r+\rho)|\, d\rho\right]^2$ can be expressed as a product of two integrals, one with respect to ρ and the other with respect to ρ'. Since $u(r+\rho)$ is square integrable with respect to ρ for all $r \ge 0$, the order of intgration may be interchanged to give

$$
\begin{aligned}
\|Tu\|^2 \;\le\; & \frac{1}{\gamma^2}\int_0^\infty \int_0^\infty K_0(\rho)\, K_0(\rho')\left[\int_0^\infty \{|u(|r-\rho|)\, u(|r-\rho'|)|\right. \\
& + |u(|r-\rho|)\, u(|r+\rho'|)| + |u(r+\rho)\, u(|r-\rho'|)| \\
& \left. + |u(r+\rho)\, u(r+\rho')|\} dr\right] d\rho d\rho' \\
=\; & \frac{1}{\gamma^2}\int_0^\infty \int_0^\infty K_0(\rho)\, K_0(\rho')\left[\int_{-\infty}^\infty \{|u(|r-\rho|)\, u(|r-\rho'|)|\right. \\
& \left. + |u(|r-\rho|)\, u(|r+\rho'|)|\} dr\right] d\rho d\rho'.
\end{aligned}
$$

By the Schwarz inequality

$$
\begin{aligned}
\int_{-\infty}^\infty \{|u(|r-\rho|)\, u(|r\pm\rho'|)|\, dr &\le \left[\int_{-\infty}^\infty u^2(|r-\rho|)\, dr \cdot \int_{-\infty}^\infty u^2(|r\pm\rho'|)\, dr\right]^{1/2} \\
&= 2\, \|u\|^2 .
\end{aligned}
$$

Since $\int_0^\infty K_0(\rho)\,d\rho = \pi/2$ [8], we obtain $\|Tu\|^2 \leq \mu^2 \|u\|^2$, which gives an upper bound on the norm of T,

$$\|T\| \leq \mu. \tag{4.2}$$

Since $C \cap L^2$ is dense in L^2, T has a norm preserving extension to L^2, so that $Tu = \langle G_0, u\rangle$ for any $u \in L^2(0,\infty)$.

To show that $Tu(r)$ is a continuous function on $[0,\infty)$ whenever $u \in L^2(0,\infty)$, we use the Schwartz inequality to write

$$|Tu(r_2) - Tu(r_1)| \leq \|u\| \left[\int_0^\infty \{G_0(r', r_2) - G_0(r', r_1)\}^2 dr'\right]^{1/2}. \tag{4.3}$$

Since $\{G_0(r', r_2) - G_0(r', r_1)\}^2$ is locally integrable and decays exponentially as $r' \to \infty$, the integral on the right-hand side of (4.3) clearly tends to 0 as $r_1 \to r_2$. As $G_0(r', r)$ is real and symmetric, we have therefore proved

Lemma 5 *T is a bounded, self-adjoint linear operator from $L^2(0,\infty)$ into $L^2(0,\infty) \cap C(0,\infty)$.*

A linear operator on $L^2(0,\infty)$ is compact if it maps every bounded set in $L^2(0,\infty)$ into a compact set. Such an operator has a discrete spectrum of eigenvalues which have no finite accumulation point in the complex plane. For the sake of convenience we shall call τ, rather than τ^{-1}, an eigenvalue of T if $\tau Tu = u$ for some $u \in L^2(0,\infty)$, $u \neq 0$. If T were compact, then it would have at most a finite number of eigenvalues in the interval $(0,1)$, and, in view of theorem 1, this would lead to the conclusion that the number of bounded solutions to the boundary value problem (1.2) is finite in $0 < \lambda < 1$. But T is not compact, as may be seen by considering the image under T of the bounded sequence $\{u_n : u_n(r) = 1 \text{ on } [n, n+1], 0 \text{ otherwise}\}$.

The non-compactness of T is a consequence of the fact that the kernel $G_0(r', r)$ preserves its shape around the logarithmic singularity as $r', r \to \infty$ in $(0,\infty) \times (0,\infty)$. In fact, T acts almost as an identity operator, whose effect is to smooth out the discontinuities of $u \in L^2(0,\infty)$ while preserving its general shape. Thus the image of the sequence u_n mentioned above is a sequence of "humps" concentrated in $[n, n+1]$ whose amplitude is not diminished as $n \to \infty$, and the sequence Tu_n has no convergent subsequence. In the next section we express $G_0(r', r)$ as a sum of its singular and nonsingular parts. The first defines an operator whose spectrum lies outside $(-1,1)$, whereas the second defines a compact operator. The compact component of T determines the spectrum of T in $(0,1)$.

5. The spectrum of T in in $(0,1)$

Lemma 6

$$G_0(r',r) = \frac{1}{\pi}\sum_{\nu=-[\mu/2]}^{[\mu/2]'} K_0\left(\sqrt{r'^2+r^2-2r'r\cos 2\nu\gamma}\right) - \frac{\mu}{\pi^2}\int_0^\infty K_0\sqrt{r'^2+r^2+2r'r\cosh\rho}\frac{\sin\mu\pi}{\cosh\mu\rho-\cos\mu\pi}d\rho \tag{5.1}$$

for all $(r',r)\in(0,\infty)\times(0,\infty)$, $r'\neq r$, and $\mu\geq 1$, where $[\mu/2]$ = greatest integer which is less than or equal to $\mu/2$, and $[\mu/2]'$ = greatest integer which is less than $\mu/2$.

Proof. Since each side of (5.1) is symmetric in r' and r, it suffices to prove the equality (5.1) when $0\leq r'<r$. Using the identity (see [8])

$$\begin{aligned} I_\alpha(r')K_\alpha(r) &= \frac{1}{\pi}\int_0^\pi K_0\left(\sqrt{r'^2+r^2-2r'r\cos\rho}\right)\cos(\alpha\rho)\,d\rho \\ &-\frac{1}{\pi}\sin(\alpha\pi)\int_0^\infty e^{-\alpha\rho}K_0\left(\sqrt{r'^2+r^2+2r'r\cosh\rho}\right)d\rho, \end{aligned}$$

which is valid on $\alpha\geq 0$ and $0\leq r'\leq r$, we have

(5.2)

$$\begin{aligned} G_0(r',r) &= \frac{1}{\gamma}\sum_{\nu=0}^\infty \varepsilon_\nu I_{\nu\mu}(r')K_{\nu\mu}(r) \\ &= \frac{1}{\gamma\pi}\sum_{\nu=0}^\infty \varepsilon_\nu\int_0^\pi K_0\left(\sqrt{r'^2+r^2-2r'r\cos\rho}\right)\cos(\nu\mu\rho)\,d\rho \\ &-\frac{1}{\gamma\pi}\sum_{\nu=0}^\infty\varepsilon_\nu\sin(\nu\mu\pi)\int_0^\infty e^{-\nu\mu\rho}K_0\left(\sqrt{r'^2+r^2+2r'r\cosh\rho}\right)d\rho. \end{aligned}$$

Since the integrals converge uniformly in ν, we may interchange the order of summation and integration to obtain

(5.3)
$$G_0(r',r) = \lim_{n\to\infty} \frac{1}{\gamma\pi}\int_0^{\pi} K_0\left(\sqrt{r'^2+r^2-2r'r\cos\rho}\right)\sum_{\nu=0}^{n}\varepsilon_\nu \cos(\nu\mu\rho)\,d\rho$$
$$-\frac{2}{\gamma\pi}\int_0^{\infty} K_0\left(\sqrt{r'^2+r^2+2r'r\cosh\rho}\right)\sum_{\nu=1}^{\infty} e^{-\nu\mu\rho}\sin(\nu\mu\pi)\,d\rho.$$

When $\rho > 0$ we have

$$\sum_{\nu=1}^{\infty} e^{-\nu\mu\rho}\sin(\nu\mu\pi) = \mathrm{Im}\sum_{\nu=1}^{\infty} e^{\nu\mu(-\rho+i\pi)} = \frac{\sin\mu\pi}{2(\cosh\mu\rho-\cos\mu\pi)},$$

which is also valid at $\rho = 0$ provided μ is not an integer. Similarly

(5.4)
$$\sum_{\nu=0}^{n}\varepsilon_\nu\cos(\nu\mu\rho) = \mathrm{Re}\sum_{\nu=0}^{n}\varepsilon_\nu e^{i\nu\mu\rho} = \frac{\sin\left(n+\frac{1}{2}\right)\mu\rho}{\sin\frac{1}{2}\mu\rho}.$$

Thus the the first integral on the right-hand side of (5.3) may be expressed as

(5.5)
$$\lim_{n\to\infty}\frac{1}{\gamma\pi}\int_0^{\pi} K_0\left(\sqrt{r'^2+r^2-2r'r\cos\rho}\right)\frac{\sin\left(n+\frac{1}{2}\right)\mu\rho}{\sin\frac{1}{2}\mu\rho}d\rho$$
$$=\lim_{n\to\infty}\frac{2}{\pi^2}\int_0^{\mu\pi/2} K_0\left(\sqrt{r'^2+r^2-2r'r\cos\tfrac{2}{\mu}\rho}\right)\frac{\sin(2n+1)\rho}{\sin\rho}d\rho$$
$$=\sum_{\nu=0}^{[\mu/2]-1}\lim_{n\to\infty}\frac{2}{\pi^2}\int_{\nu\pi}^{(\nu+1)\pi} K_0\left(\sqrt{r'^2+r^2-2r'r\cos\tfrac{2}{\mu}\rho}\right)\frac{\sin(2n+1)\rho}{\sin\rho}d\rho$$
$$+\lim_{n\to\infty}\frac{2}{\pi^2}\int_{[\mu/2]\pi}^{\mu\pi/2} K_0\left(\sqrt{r'^2+r^2-2r'r\cos\tfrac{2}{\mu}\rho}\right)\frac{\sin(2n+1)\rho}{\sin\rho}d\rho$$
$$=\sum_{\nu=0}^{[\mu/2]-1}\lim_{n\to\infty}\frac{2}{\pi^2}\int_0^{\pi} K_0\left(\sqrt{r'^2+r^2-2r'r\cos\tfrac{2}{\mu}(\rho+\nu\pi)}\right)\frac{\sin(2n+1)\rho}{\sin\rho}d\rho+$$
$$\lim_{n\to\infty}\frac{2}{\pi^2}\int_0^{\left(\frac{\mu}{2}-\left[\frac{\mu}{2}\right]\right)\pi} K_0\left(\sqrt{r'^2+r^2-2r'r\cos\tfrac{2}{\mu}(\rho+[\mu/2]\,\pi)}\right)\frac{\sin(2n+1)\rho}{\sin\rho}d\rho.$$

This is a sum of Dirichlet integrals which may be evaluated by the following formula:

$$\lim_{n\to\infty}\int_0^{a\pi} f(\rho)\frac{\sin(2n+1)\rho}{\sin\rho}d\rho = \begin{cases} \frac{\pi}{2} f(0) & \text{if } 0 < a < 1, \\ \frac{\pi}{2}\left[f(0)+f(\pi)\right] & \text{if } a = 1, \end{cases}$$

where f is a continuous function of bounded variation on $[0, a\pi]$. Thus the right-hand side of (5.5) becomes

$$\frac{1}{\pi}K_0(|r'-r|)+\frac{2}{\pi}\sum_{\nu=0}^{[\mu/2]-1} K_0\left(\sqrt{r'^2+r^2-2r'r\cos 2\nu\gamma}\right)$$
$$+\frac{\varepsilon(\mu)}{\pi}K_0\left(\sqrt{r'^2+r^2-2r'r\cos 2[\nu/2]\gamma}\right),$$

where $\varepsilon(\mu) = 1$ if $[\mu/2] = \mu/2$ and $\varepsilon(\mu) = 2$ if $[\mu/2] < \mu/2$. This, together with (5.3) and (5.4) proves lemma 6. □

Remark. It is worth Noting that the term represented by the integral in (5.1) vanishes when μ is an integer, giving the expected result

$$G_0(r', r) = \frac{1}{\pi}\sum_{\nu=0}^{\mu-1} K_0\left(\left|r' - re^{i2\nu\gamma}\right|\right),$$

for then Green's function becomes a finite sum which can be constructed by the method of images. That term, therefore, gives the contribution of the fractional part of μ, that is $\mu - [\mu]$. It carries the sign of $-\sin\mu\pi$, which is $(-1)^{[\mu]+1}$.

If τ is an eigenvalue of T corresponding to the eigenfunction $u \in L^2$, then $|\tau^{-1}| = \|Tu\| / \|u\| \leq \mu$ by (4.2), and we obtain a lower bound on $|\tau|$:

$$|\tau| \geq \frac{1}{\mu} \text{ for any } \gamma \in (0,\pi). \tag{5.6}$$

Since $G_0(r', r)$ depends on μ, the operator T and its eigenvalues will be functions of μ. We shall indicate this dependence, when it becomes important to do so, by writing $T(\mu)$, $G_0(r', r, \mu)$ and $\tau(\mu)$. For the special angle $\gamma = \pi/2$ we have the sharper result:

Lemma 7 $\|T(2)\| \leq 1$.

Proof. Let $u \in L^2(0,\infty)$ be arbitrary, and let $h(r) = 1$ on $r \geq 0$, 0 on $r < 0$. Then

$$\begin{aligned}
\langle T(2)u,u\rangle &= \int_0^\infty \int_0^\infty G_0(r',r,2)\,u(r')\,u(r)\,dr'dr \\
&= \frac{1}{\pi}\int_0^\infty \int_0^\infty [K_0(|r'-r|) + K_0(r'+r)]\,u(r')\,u(r)\,dr'dr \\
&= \frac{1}{\pi}\int_0^\infty u(r)\left[\int_0^\infty K_0(\rho)\,h(r-\rho)\,u(r-\rho)\,d\rho \right.\\
&\left. + \int_0^\infty K_0(\rho)\,u(\rho+r)\,d\rho\right]dr \\
&+ \frac{1}{\pi}\int_0^\infty u(r)\int_0^\infty K_0(\rho)\,h(\rho-r)\,u(\rho-r)\,d\rho dr.
\end{aligned}$$

By interchanging the order of integration,

$$\begin{aligned}
\langle T(2)u,u\rangle &= \frac{1}{\pi}\int_0^\infty K_0(\rho)\left[\int_0^\infty u(r)\,u(|r-\rho|)\,dr\right.\\
&\left. + \int_0^\infty u(r)\,u(r+\rho)\,dr\right]d\rho \\
&= \frac{1}{\pi}\int_0^\infty K_0(\rho)\int_{-\infty}^\infty u(|r|)\,u(|r-\rho|)\,drd\rho,
\end{aligned}$$

and applying the Schwartz inequality,

$$\left|\int_{-\infty}^\infty u(|r|)\,u(|r-\rho|)\,dr\right| \leq \sqrt{\int_{-\infty}^\infty u^2(|r|)\,dr \cdot \int_{-\infty}^\infty u^2(|r-\rho|)\,dr} = 2\,\|u\|^2,$$

we obtain

$$|\langle T(2)u,u\rangle| \leq \frac{2\,\|u\|^2}{\pi}\int_0^\infty K_0(\rho)\,d\rho = \|u\|^2. \tag{5.7}$$

Therefore

$$\|T(2)\| = \sup_{\|u\|=1} |\langle T(2)u,u\rangle| \leq 1.$$

□

Lemma 8 *$G_0(r',r,\mu) < G_0(r',r,2)$ for all $\mu \in [1,2)$ and $r' \neq r$.*

Proof. Since

$$G_0(r', r, 1) = \frac{1}{\pi} K_0(|r' - r|) < \frac{1}{\pi} K_0(|r' - r|) + \frac{1}{\pi} K_0(r' + r) = G_0(r', r, 2),$$

we need only consider the case where $1 < \mu < 2$. Defining $\mu' = \mu - 1 \in (0,1)$, and using equation (5.1), we have

$$\begin{aligned} &G_0(r', r, \mu) - G_0(r', r, 2) = \\ &\quad -\frac{\mu}{\pi^2} \int_0^\infty K_0 \sqrt{r'^2 + r^2 + 2r'r \cosh \rho} \frac{\sin \mu\pi}{\cosh \mu\rho - \cos \mu\pi} d\rho - \frac{1}{\pi} K_0(r' + r) \\ &\quad = \frac{1}{\pi^2} \int_0^\infty K_0 \sqrt{r'^2 + r^2 + 2r'r \cosh(\rho/\mu)} \frac{\sin \mu'\pi}{\cosh \rho + \cos \mu'\pi} d\rho - \frac{1}{\pi} K_0(r' + r) \\ &\quad \le \frac{1}{\pi^2} K_0(r' + r) \int_0^\infty \frac{\sin \mu'\pi}{\cosh \rho + \cos \mu'\pi} d\rho - \frac{1}{\pi} K_0(r' + r), \end{aligned}$$

since K_0 is a monotonically decreasing function over $(0, \infty)$ and $\frac{\sin \mu'\pi}{\cosh \rho + \cos \mu'\pi} \ge 0$ over $(0, \infty)$ for every $\mu' \in (0,1)$. Using the formula (see [3])

$$\int_0^\infty \frac{\sin \mu'\pi}{\cosh \rho + \cos \mu'\pi} d\rho = \mu'\pi,$$

we obtain

$$G_0(r', r, \mu) - G_0(r', r, 2) \le (\mu' - 1) \frac{1}{\pi} K_0(r' + r) < 0$$

for all $\mu' \in (0,1)$, and $r', r \in (0, \infty)$. □

The more general inequalty

$$G_0(r', r, \mu) < G_0(r', r, m), \tag{5.8}$$

where $m - 1 \le \mu < m$, m is any positive integer, and $r' \ne r$, can also be proved. When m is even, essentially the same technique used above shows that the contribution of $\mu - (m-1)$ to $G_0(r', r, \mu)$, represented by the integral in (5.1), is less than $\frac{1}{\pi} K_0(r' + r)$, the first term in the series representing $G_0(r', r, m)$. When m is odd, $G_0(r', r, \mu) - G_0(r', r, m)$ coincides with the integral on the right-hand side of (5.1), which in this case is negative since $\sin \mu\pi > 0$.

We now define the bounded and self-adjoint linear operators T_1 and T_2 on $L^2(0,\infty)$ by

$$T_1 u(r) = \int_0^\infty \frac{1}{\pi} K_0(|r' - r|)\, u(r')\, dr',$$
$$T_2 u(r) = \int_0^\infty \left[G_0(r', r) - \frac{1}{\pi} K_0(|r' - r|) \right] u(r')\, dr',$$

so that $T = T_1 + T_2$. Since K_0 is positive on $(0,\infty)$, we can write

(5.9)
$$\begin{aligned}
\|T_1\| &\leq \sup_{\|u\|=1} \int_0^\infty \int_0^\infty \frac{1}{\pi} K_0(|r' - r|)\, |u(r')\, u(r)|\, dr' dr \\
&< \sup_{\|u\|=1} \int_0^\infty \int_0^\infty \frac{1}{\pi} |K_0(|r' - r|) + K_0(|r' + r|)|\, |u(r')\, u(r)|\, dr' dr \leq 1,
\end{aligned}$$

where the last inequalty follows from the proof of Lemma 7. Thus the spectrum of T_1 lies outside the closed interval $[-1, 1]$. To prove that T_2 is compact, it suffices to show that its kernel is square integrable on $(0,\infty) \times (0,\infty)$, i.e., that T_2 is a Hilbert-Schmidt operator. For any fixed angle $\gamma \in (0, \pi)$ there is a positive integer m such that $1 \leq m - 1 \leq \mu = \pi/\gamma < m$. By Lemma 6 and equation (5.8),

$$\begin{aligned}
G_0(r', r, \mu) - \frac{1}{\pi} K_0(|r' - r|) &< G_0(r', r, m) - \frac{1}{\pi} K_0(|r' - r|) \\
&= \frac{1}{\pi} \sum_{\nu=1}^{m-1} K_0\left(\sqrt{r'^2 + r^2 - 2r'r\cos(2\nu\pi/m)} \right).
\end{aligned}$$

In terms of the variables $\rho = \sqrt{r'^2 + r^2} \in (0,\infty)$ and $\alpha = \tan^{-1}(r'/r) \in (0, \pi/2)$, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty K_0^2\left(\sqrt{r'^2 + r^2 - 2r'r\cos(2\nu\pi/m)} \right) dr' dr \\
&\qquad = \int_0^\infty \int_0^{\pi/2} K_0^2\left(\rho\sqrt{1 - \sin 2\alpha \cos(2\nu\pi/m)} \right) \rho\, d\alpha\, d\rho \\
&\qquad \leq \frac{\pi}{2} \int_0^\infty K_0^2\left(\rho\sqrt{1 - \cos(2\nu\pi/m)} \right) \rho\, d\rho \\
&\qquad \leq \frac{\pi}{2} \int_0^\infty K_0^2\left(\rho\sqrt{1 - \cos(2\nu\pi/m)} \right) \rho\, d\rho < \infty.
\end{aligned}$$

Thus the spectrum of T_2 consists of at most a countable sequence of real eigenvalues of finite multiplicities, which may be arranged in an increasing order of magnitude, with ∞ as the only point of accumulation. Let $0 < \tau_1 < \tau_2 < \tau_3 < \cdots$ be the positive part of the spectrum of T_2. According to a theorem due to H. Weyl (see [11]), the addition of a compact, self-adjoint operator to a self-adjoint operator does not change the limit points of the spectrum the latter. That means T can only have discrete eigenvalues in the interval $(0, 1)$, which will be denoted by $\{0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots\}$. This set may be empty, but the following theorem shows that it is not infinite.

Theorem 2 *The spectrum of T in $(0,1)$ consists of at most a finite number of eigenvalues.*

Proof. In view of the inequality (5.9), we may write $\|T_1\| = 1 - \delta$ for some positive number δ less than 1. By the maximum property of eigenvalues [1], the sequence $\{\lambda_i\}$ is characterized by the following inequalities:

$$\begin{aligned}
\lambda_1^{-1} &= \sup_{\|u\|=1} \langle Tu, u\rangle \leq \sup_{\|u\|=1} \langle T_1 u, u\rangle + \sup_{\|u\|=1} \langle Tu_2, u\rangle = \|T_1\| + \tau_1^{-1} \\
&= 1 - \delta + \tau_1^{-1}, \\
\lambda_2^{-1} &\leq 1 - \delta + \sup_{\|u\|=1, u \perp X_1} \langle Tu_2, u\rangle \\
&= 1 - \delta + \tau_2^{-1}, \\
&\vdots \\
\lambda_i^{-1} &\leq 1 - \delta + \sup_{\|u\|=1, u \perp X_{i-1}} \langle Tu_2, u\rangle \\
&= 1 - \delta + \tau_i^{-1}, \\
&\vdots,
\end{aligned}$$

where X_i is the subspace of $L^2(0, \infty)$ spanned by the eigenfunctions of $\lambda_1, \lambda_2, \cdots, \lambda_i$. If the sequence $\{\tau_i\}$ is finite, then clearly so is $\{\lambda_i\} \cap (0, 1)$. If $\{\tau_i\}$ is infinite, then $\tau_i \to \infty$ as $i \to \infty$, so there is an n such that $\tau_{n+1}^{-1} \leq \delta < \tau_n^{-1}$, which implies $\lambda_i \geq 1$ for all $i > n$. □

From the proof of Theorem 2 we conclude that an upper bound on the number of eigenvalues of T in $(0, 1)$ is given by the number of eigenvalues of the compact operator T_2 which are less than $(1 - \|T_1\|)^{-1}$. Let $\Lambda_n = \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ be the set of eigenvalues of T in $(0, 1)$.

Theorem 3 *$T(\mu)$ has no eigenvalues in $(0,1)$ when $1 \le \mu \le 2$.*

Proof. For any $u \in L^2(0,\infty)$ and $\mu \in [1,2]$,

$$\begin{aligned} |\langle T(\mu) u, u\rangle| &\le \int_0^\infty \int_0^\infty G_0(r', r, \mu) |u(r') u(r)| \, dr' dr \\ &\le \int_0^\infty \int_0^\infty G_0(r', r, 2) |u(r') u(r)| \, dr' dr \\ &\le \|u\|^2 \end{aligned}$$

by (5.7). Hence $\|T(\mu)\| \le 1$ and $|\lambda(\mu)| \ge \|T(\mu)\|^{-1} \ge 1$. □

Thus Λ_n is empty when $1 \le \mu \le 2$. The existence of Stokes' edge wave and its associated eigenvalue $\lambda = \sin\gamma$ guarantees that Λ_n is not empty on $\mu > 2$.

6. Solutions of the integral equation on $0 < \lambda < 1$

From Lemma 4 we know that the solutions in W of the boundary value problem (1.2) over the frequency parameter range $0 < \lambda < 1$ are of order $e^{-c(\lambda)r}$ as $r \to \infty$, where $c(\lambda) = \min\left\{\sqrt{1-\lambda^2}, b\sin\lambda\right\}$ is positive. By Theorem 1 the same is true of the solutions in W_0 of the integral equation

$$\psi(r) = -BK_0(r) + \lambda \int_0^\infty G_0(r', r)\, \psi(r')\, dr'. \tag{6.1}$$

When $\mu > 2$, Λ_n is not empty and we define $\kappa_\nu = \frac{1}{\nu}(1 - \lambda_n) > 0$ for positive values of the integer ν, λ_n being the greatest eigenvalue in Λ_n. If $0 < \lambda \le 1 - \kappa_\nu$, then $c(\lambda) \ge c(1-\kappa_\nu)$. Let W_ν denote the subset of functions in W_0 which decay like $e^{-c(1-\kappa_\nu)r}$ as $r \to \infty$. It follows that W_ν contains all the solutions in W_0 of equation (6.1) when $0 < \lambda \le 1 - \kappa_\nu$. Since $W_\nu \subset L^2(0,\infty)$ for every positive integer ν, the integral equation (6.1) may be expressed as

$$\psi(r) = -BK_0(r) + \lambda T\psi(r). \tag{6.2}$$

If ψ is bounded at $r = 0$, (6.2) is reduced to the homogeneous equation

$$\psi(r) = \lambda T\psi(r). \tag{6.3}$$

For $0 < \lambda \leq 1 - \kappa_\nu$ and $\psi \in W_0 \cap L^2(0,\infty)$, equation (6.3) has a solution only if $\lambda \in \Lambda_n$. Now κ_ν may be made arbitrarily small by taking ν large enough, but no new eigenvalues for (6.3) will appear in the process, since $1 - \kappa_1 = \lambda_n$ is the greatest eigenvalue in $(0,1)$ of that equation. Thus ψ is a solution in W_0 of the integral equation (6.1) with $B = 0$ only if the corresponding frequency parameter is in Λ_n. Conversely, we can show that every eigenfunction of T in $L^2(0,\infty)$ corresponding to an eigenvalue in Λ_n also belongs to W_0. The continuity of ψ follows from Lemma 5, so it suffices to show that $\psi \in C^1(0,\infty)$. Recalling the properties of the kernel function G_0, we now observe that the improper integrals

$$\int_0^\infty G_0(r', r)\, \psi(r')\, dr', \quad \int_0^\infty \tfrac{\partial}{\partial r} G_0(r', r)\, \psi(r')\, dr'$$

both converge uniformly over $0 < r < \infty$, hence we can write

$$\psi'(r) = \int_0^\infty \tfrac{\partial}{\partial r} G_0(r', r)\, \psi(r')\, dr',$$

which is continuous on $(0,\infty)$.

We therefore conclude that the homogeneous solutions in W_0 of the integral equation (6.1) are precisely the Λ_n eigenfunctions of T in $L^2(0,\infty)$. From Lemma 2 and the subsequent remark, we know that any bounded solution of the boundary value problem (1.2) is unique. Therefore, by Theorem 1, so are the above eigenfunctions, that is, each eigenvalue in Λ_n has multiplicity 1.

The singular solutions of (1.2) are precisely the nonhomogeneous solutions of (6.1). The existence of such solutions would have been guaranteed for every $\lambda \in (0,1) \setminus \Lambda_n$ by the Fredholm alternative theorem had T been compact [11]. Under the circumstances, we consider the transformed equation

$$\psi(r) - \lambda (I - \lambda T_1)^{-1} T_2 \psi(r) = -B (I - \lambda T_1)^{-1} K_0(r), \tag{6.4}$$

which is obtained from (6.2) by expressing T as $T_1 + T_2$ and multiplying by $(I - \lambda T_1)^{-1}$. Since

$$\|I - \lambda T_1\| \geq 1 - \lambda \|T_1\| = 1 - \lambda (1 - \delta) \geq \delta$$

for all $0 < \lambda \leq 1$, the operator $(I - \lambda T_1)^{-1}$ exists and is bounded on $L^2(0,\infty)$ for all values of λ in $[0,1]$. Thus equations (6.2) and (6.4) are equivalent in the sense that ψ is a soluton of one if, and only if, it is a solution of the other. Since the product of a bounded operator and a compact operator is compact, $(I-\lambda T_1)^{-1} T_2$ is compact provided $0 \leq \lambda \leq 1$. Now equation (6.3) has only the trivial solution for every $\lambda \in (0,1) \setminus \Lambda_n$, so the same applies to the homogeneous equation which corresponds to (6.4), that is $\psi(r) - \lambda (I-\lambda T_1)^{-1} T_2 \psi(r) = 0$, the two equations being equivalent. Therefore, by the Fredholm alternative theorem, equation (6.4) has a solution for every $\lambda \in (0,1) \setminus \Lambda_n$. Moreover, when $\lambda = \lambda_i \in \Lambda_n$ is an eigenvalue of T associated with the eigenfunction ψ_i, equation (6.4) is solvable if, and only if, $\langle \psi_i, K_0 \rangle = \left\langle \psi_i^*, (I-\lambda T_1)^{-1} K_0 \right\rangle = 0$, where ψ_i^* is the solution which corresponds to ψ_i of the adjoint equation

$$\psi^*(r) - \lambda T_2 (I-\lambda T_1)^{-1} \psi^*(r) = 0.$$

But we have

$$\begin{aligned}
\langle \psi_i, K_0 \rangle &= \int_0^\infty K_0(r)\, \psi_i(r)\, dr \\
&= \gamma \int_0^\infty G_0(0,r)\, \psi_i(r)\, dr \\
&= \gamma T \psi_i(0) \\
&= \frac{\gamma}{\lambda_i} \psi_i(0) \neq 0
\end{aligned}$$

by Lemma 2. Consequently equation (6.4), and hence (6.2), has no solution when $\lambda \in \Lambda_n$ and $B \neq 0$.

On the basis of Theorem 1 and the results of sections 5 and 6, we may now sum up our findings:

Case 1. $\pi/2 \leq \gamma < \pi$

(i) The boundary value problem (1.2) has no bounded solution.

(ii) A unique logarithmic logarithmic solution exists at every value of $\lambda \in (0,1)$.

Case 2. $0 < \gamma < \pi/2$

(i) The boundary value problem (1.2) has at least one and at most finitely many bounded solutions, each corresponding uniquely to a discrete value of the frequency parameter λ in the set

$$\Lambda_n = \{\lambda_i : 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < 1\},$$

where n depends on γ and $\lambda_1 \geq \gamma/\pi$. Moreover, these bounded solutions are orthogonal in the topology of $L^2(0,\infty)$, being eigenfunctions of a self-adjoint, compact operator.

(ii) For every $\lambda \in (0,1) \setminus \Lambda_n$ a unique logarithmic solution exists.

(iii) The bounded and the logarithmic solutions do not coexist for the same $\lambda \in (0,1)$.

References

[1] K.O. Friedrichs, *Spectral Theory of Operators in Hilbert Space*, New York University, 1960.

[2] I.M. Gel'fand and G.E. Shilov, *Generalized Functions*, vol. 1, Academic Press, New York, 1964.

[3] I.S. Gradshteyn and I.M. Rizhik, *Tables of Integrals, Series, and Products*, Academic Press, New York, 1965.

[4] H.A. Lauwerier, Solutions of the equation of Helmholtz in an angle, Kon. Ned. Akad. v. Wet. Proc. A, vol. 62, no. 5, 1959, 475-488, and vol. 63, no. 4, 1960, 355-372.

[5] R.S. Lehman, Developments in the neighborhood of the beach of surface waves over an inclined bottom, Comm. Pure Appl. Math., vol. 7, 1954, 393-439.

[6] R.S. Lehman and H. Lewy, Uniqueness of water waves on a sloping beach, Comm. Pure Appl. Math., vol. 14, 1961, 521-546.

[7] H. Lewy, Developments at the confluence of analytic boundary conditions, Univ. of California Publ. Math., vol. 1, 1950, 247-280.

[8] W. Magnus, F. Oberhettinger, and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966.

[9] A.S. Peters, Water waves over sloping beaches and the solution of a mixed boundary value problem for $\Delta\phi - k^2\phi = 0$ in a sector, Comm. Pure Appl. Math., vol. 5, 1952, 87-108.

[10] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, 1967.

[11] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar, New york, 1955.

[12] M. Roseau, Short waves parallel to the shore over a sloping beach, Comm. Pure Appl. Math., vol. 11, 1958.

[13] J.J. Stoker, *Water Waves*, Interscience, New York, 1957.

[14] R. Ursell, Edge waves on a sloping beach, Proc. Roy. Soc., ser. A, vol. 214, 1952, 79-97.