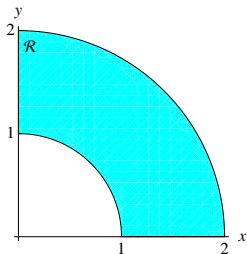


Double Integrals in Polar Coordinates

1. A flat plate is in the shape of the region \mathcal{R} in the first quadrant lying between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. The density of the plate at point (x, y) is $x + y$ kilograms per square meter (suppose the axes are marked in meters). Find the mass of the plate.



Solution. As we saw in #2(b) of the worksheet “Double Integrals”, the mass is the double integral of density. That is, the mass is $\iint_{\mathcal{R}} (x + y) \, dx \, dy$.

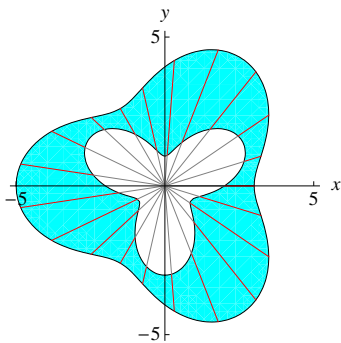
To compute double integrals, we always convert them to iterated integrals. In this case, we’ll use a double integral in polar coordinates. The region \mathcal{R} is the polar rectangle $0 \leq \theta \leq \frac{\pi}{2}$, $1 \leq r \leq 2$, so we can rewrite the double integral as an iterated integral in polar coordinates:

$$\begin{aligned}
 \iint_{\mathcal{R}} (x + y) \, dx \, dy &= \int_0^{\pi/2} \int_1^2 (r \cos \theta + r \sin \theta)(r \, dr \, d\theta) \\
 &= \int_0^{\pi/2} \int_1^2 r^2 (\cos \theta + \sin \theta) \, dr \, d\theta \\
 &= \int_0^{\pi/2} \left(\frac{1}{3} r^3 (\cos \theta + \sin \theta) \Big|_{r=1}^{r=2} \right) d\theta \\
 &= \int_0^{\pi/2} \frac{7}{3} (\cos \theta + \sin \theta) \, d\theta \\
 &= \frac{7}{3} (\sin \theta - \cos \theta) \Big|_{\theta=0}^{\theta=\pi/2} \\
 &= \boxed{\frac{14}{3}}
 \end{aligned}$$

2. Find the area of the region \mathcal{R} lying between the curves $r = 2 + \sin 3\theta$ and $r = 4 - \cos 3\theta$. (You may leave your answer as an iterated integral in polar coordinates.)

Solution. As we saw in #2(a) of the worksheet “Double Integrals”, the area of the region \mathcal{R} is equal to the double integral $\iint_{\mathcal{R}} 1 \, dx \, dy$. To compute the value of this double integral, we will convert it to an iterated integral.

This region is not a polar rectangle, so we’ll think about slicing. Let’s make slices where θ is constant:

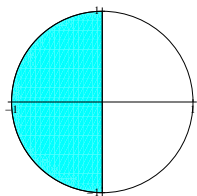


Our slices go all the way around the origin, so the outer integral will have θ going from 0 to 2π . Along each slice, r goes from the inner curve ($r = 2 + \sin 3\theta$) to the outer curve ($r = 4 - \cos 3\theta$). So, the iterated integral is

$$\begin{aligned}
 \boxed{\int_0^{2\pi} \int_{2+\sin 3\theta}^{4-\cos 3\theta} 1 \cdot r \, dr \, d\theta} &= \int_0^{2\pi} \left(\frac{1}{2} r^2 \Big|_{r=2+\sin 3\theta}^{r=4-\cos 3\theta} \right) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} [(4 - \cos 3\theta)^2 - (2 + \sin 3\theta)^2] \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (16 - 8 \cos 3\theta + \cos^2 3\theta - 4 - 4 \sin 3\theta - \sin^2 3\theta) \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (12 - 8 \cos 3\theta + \cos 6\theta - 4 \sin 3\theta) \, d\theta \\
 &\quad \text{by the double angle identity } \cos 2t = \cos^2 t - \sin^2 t \\
 &= \frac{1}{2} \left(12\theta - \frac{8}{3} \sin 3\theta + \frac{1}{6} \sin 6\theta + \frac{4}{3} \cos 3\theta \right) \Big|_{\theta=0}^{\theta=2\pi} \\
 &= \boxed{12\pi}
 \end{aligned}$$

3. In each part, rewrite the double integral as an iterated integral in polar coordinates. (Do not evaluate.)

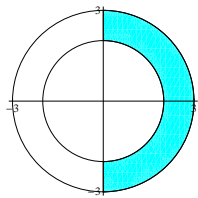
(a) $\iint_{\mathcal{R}} \sqrt{1-x^2-y^2} \, dx \, dy$ where \mathcal{R} is the left half of the unit disk.



Solution. The region \mathcal{R} is the polar rectangle $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, $0 \leq r \leq 1$. In polar coordinates, the integrand is $\sqrt{1-r^2}$. So, we can rewrite the double integral as an iterated integral

$$\boxed{\int_{\pi/2}^{3\pi/2} \int_0^1 \sqrt{1-r^2} \cdot r \, dr \, d\theta}.$$

- (b) $\iint_{\mathcal{R}} x^2 dx dy$ where \mathcal{R} is the right half of the ring $4 \leq x^2 + y^2 \leq 9$.



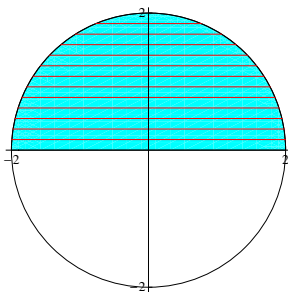
Solution. The region \mathcal{R} is the polar rectangle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $2 \leq r \leq 3$.⁽¹⁾ In polar coordinates, the integrand is $(r \cos \theta)^2$. So, we can rewrite the double integral as an iterated integral

$$\int_{-\pi/2}^{\pi/2} \int_2^3 r^2 \cos^2 \theta \cdot r dr d\theta.$$

4. Rewrite the iterated integral in Cartesian coordinates $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xy dx dy$ as an iterated integral in polar coordinates. (Try to draw the region of integration.) You need not evaluate.

Solution. Let's first write the integrand in polar coordinates. Since $x = r \cos \theta$ and $y = r \sin \theta$, the integrand can be written as $r^2 \sin \theta \cos \theta$.

Next, let's figure out the region of integration. Since the outer integral is \int_0^2 something dy , we are slicing the interval $[0, 2]$ on the y -axis, so we are making horizontal slices from $y = 0$ to $y = 2$. The inner integral tells us that the left side of each slice is on $x = -\sqrt{4-y^2}$ and the right side of each slice is on $x = \sqrt{4-y^2}$. $x = -\sqrt{4-y^2}$ is the left half of the circle $x^2 + y^2 = 4$, and $x = \sqrt{4-y^2}$ is the right half of the circle $x^2 + y^2 = 4$, so our region of integration (with horizontal slices) looks like this:

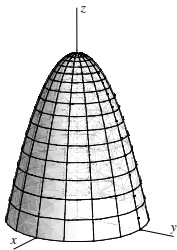


This region is the polar rectangle $0 \leq \theta \leq \pi$, $0 \leq r \leq 2$. So, the integral in polar coordinates is

$$\int_0^{\pi} \int_0^2 r^2 \sin \theta \cos \theta \cdot r dr d\theta.$$

5. Find the volume of the solid enclosed by the xy -plane and the paraboloid $z = 9 - x^2 - y^2$. (You may leave your answer as an iterated integral in polar coordinates.)

⁽¹⁾Normally, we want θ to be between 0 and 2π . However, if it's more convenient for a polar integral, we relax this restriction.



Solution. Let's break this down into two steps:

1. First, we'll write a double integral expressing the volume.
2. Then, we'll convert the double integral to an iterated integral.

Notice that the solid can be described as the solid under $z = 9 - x^2 - y^2$ over the region \mathcal{R} , where \mathcal{R} is where the solid meets the xy -plane. So, its volume will be $\iint_{\mathcal{R}} (9 - x^2 - y^2) \, dx \, dy$. Let's describe \mathcal{R} in more detail. The surface $z = 9 - x^2 - y^2$ intersects the xy -plane $z = 0$ where $x^2 + y^2 = 9$, so the region \mathcal{R} is the disk $x^2 + y^2 \leq 9$.

Now, we'll convert this double integral to an iterated integral. The region \mathcal{R} is the polar rectangle $0 \leq \theta \leq 2\pi, 0 \leq r \leq 3$, so we can rewrite the double integral as

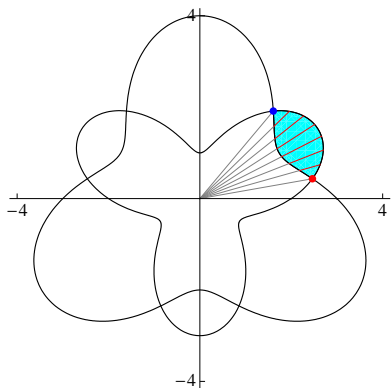
$$\begin{aligned}
 \iint_{\mathcal{R}} (9 - x^2 - y^2) \, dx \, dy &= \boxed{\int_0^{2\pi} \int_0^3 (9 - r^2)r \, dr \, d\theta} \\
 &= \int_0^{2\pi} \int_0^3 (9r - r^3) \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^3 \left(\frac{9}{2}r^2 - \frac{r^4}{4} \Big|_{r=0}^{r=3} \right) d\theta \\
 &= \int_0^{2\pi} \frac{81}{4} d\theta \\
 &= \frac{81}{4} \theta \Big|_{\theta=0}^{\theta=2\pi} \\
 &= \boxed{\frac{81\pi}{2}}
 \end{aligned}$$

6. The region inside the curve $r = 2 + \sin 3\theta$ and outside the curve $r = 3 - \sin 3\theta$ consists of three pieces. Find the area of one of these pieces. (You may leave your answer as an iterated integral in polar coordinates.)

Solution. Since we are finding area, our integral will be $\iint_{\mathcal{R}} 1 \, dx \, dy$, where \mathcal{R} is the region of integration. As always, to evaluate the double integral, we need to rewrite it as an iterated integral (this time, in polar coordinates).

Let's make slices where $\theta = \text{constant}$.⁽²⁾

⁽²⁾When we're dealing with regions that aren't polar rectangles, it's almost always easier to slice where $\theta = \text{constant}$.

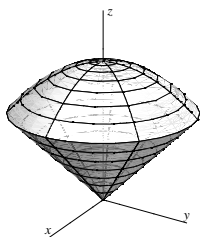


We are slicing from the θ of the red point to the θ of the blue point. Let's find these. The red point and blue point are points where the curves $r = 2 + \sin 3\theta$ and $r = 3 - \sin 3\theta$ intersect, so let's solve $2 + \sin 3\theta = 3 - \sin 3\theta$. This happens when $\sin 3\theta = \frac{1}{2}$, or $3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. So, the red point has $\theta = \frac{\pi}{18}$, the blue point has $\theta = \frac{5\pi}{18}$, and our outer integral will have θ going from $\frac{\pi}{18}$ to $\frac{5\pi}{18}$.

Along a slice, r goes from the inner curve ($r = 3 - \sin 3\theta$) to the outer curve ($r = 2 + \sin 3\theta$), so we can rewrite our double integral as

$$\begin{aligned} \int_{\pi/18}^{5\pi/18} \int_{3-\sin 3\theta}^{2+\sin 3\theta} 1 \cdot r \, dr \, d\theta &= \int_{\pi/18}^{5\pi/18} \left(\frac{1}{2} r^2 \Big|_{r=3-\sin 3\theta}^{r=2+\sin 3\theta} \right) d\theta \\ &= \frac{1}{2} \int_{\pi/18}^{5\pi/18} [(2 + \sin 3\theta)^2 - (3 - \sin 3\theta)^2] \, d\theta \\ &= \frac{1}{2} \int_{\pi/18}^{5\pi/18} (-5 + 10 \sin 3\theta) \, d\theta \\ &= \frac{1}{2} \left(-5\theta - \frac{10}{3} \cos 3\theta \right) \Big|_{\theta=\pi/18}^{\theta=5\pi/18} \\ &= \boxed{\frac{5}{\sqrt{3}} - \frac{5\pi}{9}} \end{aligned}$$

7. Find the volume of the “ice cream cone” bounded by the single cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 3 - \frac{x^2}{4} - \frac{y^2}{4}$.

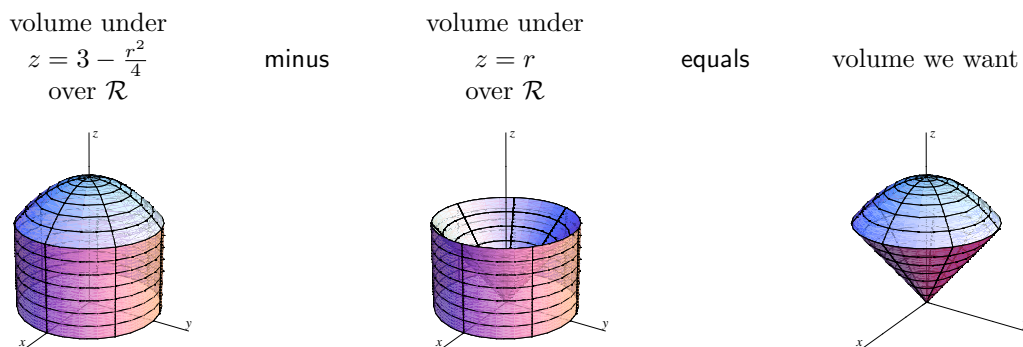


Solution. Let \mathcal{R} be the projection of the solid onto the xy -plane; that is, let \mathcal{R} be the region we see if we look down on the solid from above. This will be a disk, so let's do the integral in polar coordinates.

First, we'll rewrite everything in terms of polar coordinates. The cone $z = \sqrt{x^2 + y^2}$ can be rewritten as $z = r$, and the paraboloid $z = 3 - \frac{x^2}{4} - \frac{y^2}{4}$ can be rewritten as $z = 3 - \frac{r^2}{4}$.

To find the disk \mathcal{R} , notice that, if we look at the solid from above, the disk we see is the size of the circle where the two surfaces intersect. The surfaces intersect where $r = 3 - \frac{r^2}{4}$; this can be rewritten as $r^2 + 4r - 12 = 0$, or $(r + 6)(r - 2) = 0$. Since $r \geq 0$, the intersection is $r = 2$. So, the region \mathcal{R} is a disk centered at the origin with radius 2. This is a polar rectangle with $0 \leq r \leq 2$, $0 \leq \theta < 2\pi$.

One way to find the volume of the solid is to find the volume under the paraboloid over \mathcal{R} , find the volume under the cone over \mathcal{R} , and subtract the latter from the former.⁽³⁾ That is:



So, the iterated integral in polar coordinates is

$$\begin{aligned}
 \int_0^{2\pi} \int_0^2 \left(3 - \frac{r^2}{4}\right) r \, dr \, d\theta - \int_0^{2\pi} \int_0^2 r \cdot r \, dr \, d\theta &= \int_0^{2\pi} \int_0^2 \left(3 - \frac{r^2}{4} - r\right) r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 \left(3r - \frac{r^3}{4} - r^2\right) \, dr \, d\theta \\
 &= \int_0^{2\pi} \left(\frac{3r^2}{2} - \frac{r^4}{16} - \frac{r^3}{3} \Big|_{r=0}^{r=2}\right) d\theta \\
 &= \int_0^{2\pi} \frac{7}{3} \, d\theta \\
 &= \boxed{\frac{14\pi}{3}}
 \end{aligned}$$

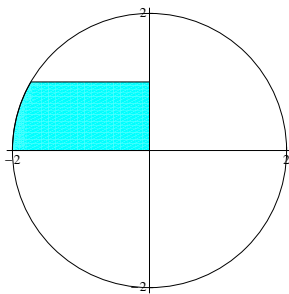
Notice that we end up simply integrating the difference between $3 - \frac{r^2}{4}$ and r ; this is really the height of the solid above the point (r, θ) . For an explanation of why this works in terms of Riemann sums, see #6 of the worksheet “Double Integrals”.

8. A flat plate is in the shape of the region \mathcal{R} defined by the inequalities $x^2 + y^2 \leq 4$, $0 \leq y \leq 1$, $x \leq 0$. The density of the plate at the point (x, y) is $-xy$. Find the mass of the plate.

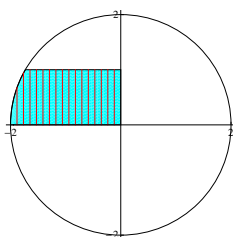
Solution. As we saw in #2(b) of the worksheet “Double Integrals”, the mass is the double integral of density. That is, the mass is $\iint_{\mathcal{R}} -xy \, dx \, dy$.

Here is a picture of the region:

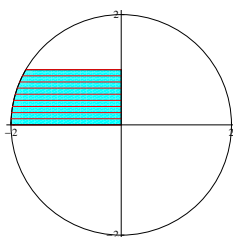
⁽³⁾This is similar to what you were asked to do in the homework problem §12.3, #30.



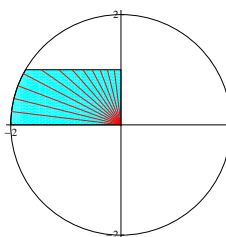
There are four ways we could slice: two in Cartesian (vertically or horizontally) and two in polar (where θ is constant or where r is constant). Here are pictures of all four:



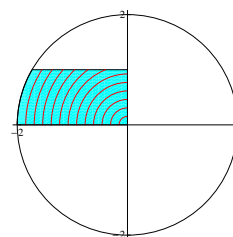
vertical ($x = \text{constant}$)



horizontal ($y = \text{constant}$)



$\theta = \text{constant}$



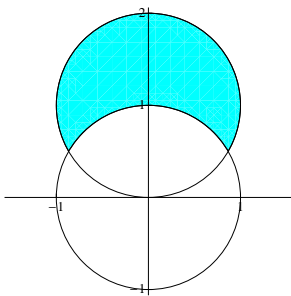
$r = \text{constant}$

When slicing vertically, along $\theta = \text{constant}$, or along $r = \text{constant}$, there are multiple “types” of slices. However, if we slice horizontally, there is only one “type” of slice. This suggests that we should go with horizontal slices. Slicing horizontally amounts to slicing the interval $[0, 1]$ on the y -axis, so the outer integral will be $\int_0^1 \text{something } dy$. Each slice has its left end on the left edge of the circle $x^2 + y^2 = 4$ (so where $x = -\sqrt{4 - y^2}$) and its right end on $x = 0$, so we can rewrite the double integral as

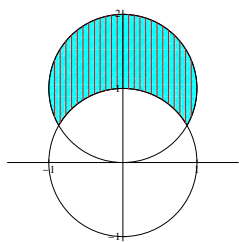
$$\begin{aligned}
 \boxed{\int_0^1 \int_{-\sqrt{4-y^2}}^0 -xy \, dx \, dy} &= \int_0^1 \left(-\frac{1}{2} x^2 y \Big|_{x=-\sqrt{4-y^2}}^{x=0} \right) dy \\
 &= \int_0^1 \frac{1}{2} (4 - y^2) y \, dy \\
 &= \frac{1}{2} \int_0^1 (4y - y^3) \, dy \\
 &= \frac{1}{2} \left(2y^2 - \frac{y^4}{4} \right) \Big|_{y=0}^{y=1} \\
 &= \boxed{\frac{7}{8}}
 \end{aligned}$$

9. Find the area of the region which lies inside the circle $x^2 + (y-1)^2 = 1$ but outside the circle $x^2 + y^2 = 1$.

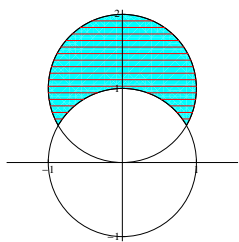
Solution. Here is a picture of the region, which we'll call \mathcal{R} :



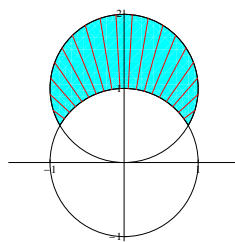
There are four ways we could slice: two in Cartesian (vertically or horizontally) and two in polar (where θ is constant or where r is constant). Here are pictures of all four:



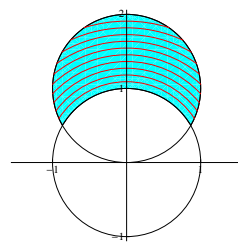
vertical ($x = \text{constant}$)



horizontal ($y = \text{constant}$)



$\theta = \text{constant}$



$r = \text{constant}$

When slicing vertically or horizontally, we can see that there are multiple “types” of slices. When slicing where $\theta = \text{constant}$ or $r = \text{constant}$, there is only one type of slice. So, let’s do this in polar coordinates.

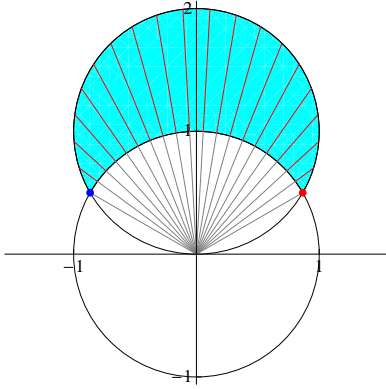
First, let’s write the equations of the two circles in polar coordinates. The circle $x^2 + y^2 = 1$ is just $r = 1$. The circle $x^2 + (y - 1)^2 = 1$ is more complicated:

$$\begin{aligned} x^2 + (y - 1)^2 &= 1 \\ (r \cos \theta)^2 + (r \sin \theta - 1)^2 &= 1 \\ r^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r \sin \theta + 1 &= 1 \\ r^2 (\cos^2 \theta + \sin^2 \theta) - 2r \sin \theta &= 0 \\ r^2 &= 2r \sin \theta \\ r &= 2 \sin \theta \end{aligned}$$

(In the last step, we’ve divided both sides by r ; this is fine since $r > 0$ on the circle.⁽⁴⁾)

We’ll use the third picture, where we slice along $\theta = \text{constant}$. (We can use the fourth as well, but we’re more used to doing polar integrals by slicing where $\theta = \text{constant}$.) Here’s a picture with more detail.

⁽⁴⁾Actually, $r = 0$ at the very bottom of the circle, but as it’s just one point, it doesn’t really matter.



We are slicing from the θ of the red point to the θ of the blue point. Let's find these values. The red point and blue point are points where the curves $r = 1$ and $r = 2 \sin \theta$ intersect, so let's solve $1 = 2 \sin \theta$. This happens when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. So, the red point has $\theta = \frac{\pi}{6}$, the blue point has $\theta = \frac{5\pi}{6}$, and our outer integral will have θ going from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$.

Along each slice, r goes from the lower circle ($r = 1$) to the upper circle ($r = 2 \sin \theta$), so the inner integral will have r going from 1 to $2 \sin \theta$. So, we can rewrite our double integral as

$$\begin{aligned}
 \boxed{\int_{\pi/6}^{5\pi/6} \int_1^{2 \sin \theta} 1 \cdot r \, dr \, d\theta} &= \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} r^2 \Big|_{r=1}^{r=2 \sin \theta} \right) d\theta \\
 &= \int_{\pi/6}^{5\pi/6} \left(2 \sin^2 \theta - \frac{1}{2} \right) d\theta \\
 &= \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} - \cos 2\theta \right) d\theta \\
 &\quad \text{by the identity } \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) \\
 &= \left. \frac{\theta}{2} - \frac{1}{2} \sin 2\theta \right|_{\theta=\pi/6}^{\theta=5\pi/6} \\
 &= \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}
 \end{aligned}$$