③ Joint distributed random variables:

let X and Y two random variables.

Discrete

Mass function:

$$f_{(X,Y)}(x,y) = P(X=x, Y=y).$$

$$\begin{cases} f_X(x) = \sum_y f_{(X,Y)}(x,y). \\ f_Y(y) = \sum_x f_{(X,Y)}(x,y) \end{cases}$$

, marginal distributions.

Conditional distribution of X given Y = y:

$$f_{X|Y=y}(x) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)}$$

Covariance of X and Y:

$$Covar(X,Y) = E(XY) - E(X)E(Y).$$

$$covar(X,X) = var(X).$$

Continuous

Density function:

* $f_{(X,Y)}(x,y) \geq 0$.
* $\iint f_{(X,Y)}(x,y)\,dx\,dy = 1$.

$$f_X(x) = \int f_{(X,Y)}(x,y)\,dy.$$

$$f_Y(y) = \int f_{(X,Y)}(x,y)\,dx.$$

Conditional distribution of X given Y = y:

$$f_{X|Y=y}(x) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)}.$$

Example(1): Let the mass function $f_{(X,Y)}$:

* $f_X = ?$
* $f_Y = ?$
* $f_{X|Y=0} = ?$
* $f_{Y|X=2} = ?$
* $E(X) =$; $E(Y) =$
* $E(XY) =$
* $Cov(X,Y) =$

Y \ X	0	1	2	f_Y	$f_{Y\|X=2}$
−1	$\frac{2}{20}$	$\frac{1}{20}$	$\frac{4}{20}$	$7/20$	$4/7$
0	$3/20$	$\frac{5}{20}$	0	$8/20$	0
1	$\frac{1}{20}$	$\frac{1}{20}$	$3/20$	$5/20$	$3/7$
f_X	$\frac{6}{20}$	$\frac{7}{20}$	$\frac{7}{20}$		
$f_{X\|Y=0}$	$3/8$	$5/8$	0		
$f_{X\|Y=1}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{5}$		

$$E(X) = (0)\frac{6}{20} + (1)\frac{7}{20} + (2)\frac{7}{20} = \frac{21}{20}.$$

$$E(Y) = (-1)\frac{7}{20} + (0)\frac{8}{20} + (1)\frac{5}{20} = \frac{-2}{20}.$$

$$E(XY) = \sum_{x,y} xy\, f_{(X,Y)}(x,y) = 0+0+0+\frac{-1}{20}+0+\frac{1}{20}+\frac{-8}{20}+0+\frac{6}{20}$$

$$E\, g(X,Y) = \sum_{x,y} g(x,y)\, f_{(X,Y)}(x,y). \qquad = \frac{-2}{20} = \frac{-1}{10}.$$

$$Cov(X,Y) = \frac{-1}{10} - \left(\frac{21}{20}\right)\left(\frac{-2}{20}\right) = \frac{-1}{10} + \frac{42}{400} = \frac{2}{400}.$$

* $E(X \mid Y=0) = (0)\cdot\frac{3}{8} + (1)\frac{5}{8} + (2)\,0 = 5/8.$

Independence:

* A and B are independent if:

$$P(A\cap B) = P(A)\cdot P(B).$$

* X and Y are independent if:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \Leftrightarrow f_{X|Y} = f_X \Leftrightarrow f_{Y|X} = f_Y.$$

$$\Leftrightarrow E\, f(X) g(Y) = (E f(X))(E g(Y)).$$

Example: let the joint density function:

$$f_{(X,Y)}(x,y) = \begin{cases} c(x + x^2 y); & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

a) $c = ?$

b) f_X

c) f_Y

d) $f_{X|Y=y}$

e) $E(X)$

f) $E(X|Y)$.

d') $f_{Y|X=x}$

e') $E(Y)$

f') $E(Y|X)$.

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a) $1 = \iint f(x,y)\, dx\, dy = \int_0^1 dy \int_0^1 dx\; c(x + x^2 y).$

$$= \int_0^1 dy\; c\left(\frac{x^2}{2} + \frac{x^3}{3} y\right)\Big|_0^1 = \int_0^1 dy \left[c\left(\frac{1}{2} + \frac{1}{3} y\right) - 0\right].$$

$$= c\left(\frac{1}{2} y + \frac{1}{6} y^2\right)\Big|_0^1 = c\left(\frac{1}{2} + \frac{1}{6}\right) - 0 = c\,\frac{4}{6} \Leftrightarrow \boxed{c = \frac{6}{4}}$$

b) $f_X(x) = \int_0^1 c(x + x^2 y)\,dy = c(xy + \frac{1}{2}x^2 y^2)\Big|_0^1$

$= c\left(x + \frac{x^2}{2}\right) = \frac{6}{4}\left(x + \frac{x^2}{2}\right) \quad 0 \le x \le 1.$

$$f_X(x) = \begin{cases} \frac{6}{4}\left(x + \frac{x^2}{2}\right), & 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

c) $f_Y(y) = \int_0^1 dx\ c(x + x^2 y)\,dx = c\left(\frac{x^2}{2} + \frac{x^3}{3}y\right)\Big|_0^1$

$= c\left(\frac{1}{2} + \frac{1}{3}y\right)$

$$f_Y(y) = \begin{cases} \frac{6}{4}\left(\frac{1}{2} + \frac{1}{3}y\right), & 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

d) $$f_{X|Y=y}(x) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)} \qquad y \in [0,1].$$

$$= \begin{cases} \dfrac{\cancel{c}(x + x^2 y)}{\cancel{c}\left(\frac{1}{2} + \frac{1}{3}y\right)} & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

e) $E(X) = \int x\, f_X(x)\,dx = \int_0^1 \frac{6}{4}x\left(x + \frac{x^2}{2}\right)dx$

$= \frac{6}{4}\int_0^1 \cancel{6}\left(x^2 + \frac{x^3}{2}\right)dx = \frac{6}{4}\left(\frac{x^3}{3} + \frac{x^4}{8}\right)\Big|_0^1$

$= \frac{6}{4}\left(\frac{1}{3} + \frac{1}{8}\right) - 0 = \frac{6}{4} \cdot \frac{11}{24} = \frac{11}{16}.$

f) $E(X|Y=y) = \int x\, f_{X|Y=y}(x)\,dx = \int_0^1 6x\, \frac{x + x^2 y}{3 + 2y}\,dx.$

$= \frac{6}{3+2y}\int_0^1 (x^2 + x^3 y)\,dx = \frac{6}{3+2y}\left(\frac{x^3}{3} + \frac{x^4}{4}y\right)\Big|_0^1$

$= \frac{6}{3+2y}\left(\frac{1}{3} + \frac{1}{4}y\right) = \frac{4+3y}{2(3+2y)}$

$E(X|Y) = \frac{4+3Y}{2(3+2Y)}.$

$$E(XY) = \iint xy\, f_{(X,Y)}(x,y)\, dx\, dy$$

$$= \int_0^1 dy \int_0^1 dx\ xy\, c(x + x^2 y)$$

$$= c\int_0^1 dy \int_0^1 dx\ (x^2 y + x^3 y^2)$$

$$= c\int_0^1 dy \left(\frac{x^3}{3} y + \frac{x^4}{4} y^2\right)\Big|_0^1$$

$$= c\int_0^1 dy \left(\frac{1}{3} y + \frac{1}{4} y^2\right) = c\left(\frac{y^2}{6} + \frac{y^3}{12}\right)\Big|_0^1$$

$$= c\left(\frac{1}{6} + \frac{1}{12}\right) = c \cdot \frac{1}{4} = \frac{6}{16} = 3/8$$

d') $$f_{Y|X=x}(y) = \begin{cases} \dfrac{c(x + x^2 y)}{c(x + x^2/2)} & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

e') $$E(Y) = \int y\, f_Y(y)\, dy = \int_0^1 cy\left(\frac{1}{2} + \frac{1}{3} y\right) dy$$

$$= c\int_0^1 \left(\frac{1}{2} y + \frac{1}{3} y^2\right) dy$$

$$= c\left(\frac{1}{4} y^2 + \frac{1}{9} y^3\right)\Big|_0^1 = c\left(\frac{1}{4} + \frac{1}{9}\right) = \frac{6}{4} \cdot \frac{13}{36}$$

$$= \frac{13}{24}$$

f') $$E(Y|X=x) = \int y\, f_{Y|X=x}(y)\, dy$$

$$= \int_0^1 y\, \frac{2(x + x^2 y)}{2x + x^2}\, dy$$

$$= \frac{2}{2x + x^2} \int_0^1 (xy + x^2 y^2)\, dy$$

$$= \frac{2}{2x + x^2} \left(x y^2/2 + x^2 y^3/3\right)\Big|_0^1$$

$$= \frac{2}{2x + x^2} \left(x/2 + x^2/3\right) = \frac{3x + 2x^2}{3(2x + x^2)}$$

Moment generating function:

let a random variable X, we define:

* the <u>moment</u> generating function ϕ_X by:

$$\phi_X(t) = E\, e^{tX}.$$

* the rth moment of X: $E(X)^r$

$$\phi_X'(t) = E\, X e^{tX} \longrightarrow \phi_X'(0) = E(X).$$

$$\phi_X''(t) = E\, X^2 e^{tX} \longrightarrow \phi_X''(0) = E(X^2).$$

$$\vdots$$

$$\phi_X^{(n)}(0) = E(X^n).$$

Example: Compute the ~~gene~~ m.g.f for:

① $X \sim Bin(p, n)$; $f_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$.

② $X \sim Exp(\lambda)$; $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0. \\ 0 & x < 0 \end{cases}$

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① $\phi_X(t) = E\, e^{tX} = \sum_{k=0}^{n} e^{tk} f_X(k).$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} = (pe^t + 1 - p)^n.$$

$\left[(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} : \text{Binomial formula} \right]$

$$\phi_X'(t) = n(pe^t + 1 - p)^{n-1} pe^t \; ; \; \phi_X'(0) = np = E(X)$$

$$\phi_X''(t) = n\cancel{p}\left[(n-1)(pe^t + 1 - p)^{n-2} (pe^t)^2 + (pe^t + 1 - p)^{n-1} pe^t \right]$$

$$\phi_X''(0) = n\cancel{p}\left[(n-1)p^2 + p \right] = E(X^2).$$

$$Var(X) = \cancel{np}\left(\cancel{(n-1)p^2} + p \right) - \cancel{n^2p^2} = np - np^2 = np(1-p).$$

② $\phi_X(t) = E\, e^{tX} = \int_0^\infty \lambda e^{tx} e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx.$

$\boxed{t < \lambda}$

$$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_0^\infty = \frac{\lambda}{\lambda - t}.$$

$$\phi_X'(t) = +\frac{\lambda}{(\lambda - t)^2} \; ; \; \phi_X'(0) = 1/\lambda = E(X).$$

$$\phi_X''(t) = +\frac{2(\lambda)}{(\lambda - t)^3} \; ; \; \phi_X''(0) = \frac{2}{\lambda^2}, \; Var(X) = \frac{1}{\lambda^2}.$$

Example: Find the moment generating function for:

① $X \sim geo(p)$; $f_X(x) = q^{x-1}p$, $q = 1-p$; $x = 1, 2, \ldots$

② $X \sim$ ~~Exp(λ)~~ $Unif((0,1))$.

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①
$$\phi_X(t) = E e^{tX} = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p \qquad \left(\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} ; |r|<1 \right)$$

$$= p q^{-1} \sum_{x=1}^{\infty} (e^t q)^x = p q^{-1} \frac{e^t q}{1 - e^t q}$$

$$\phi_X(t) = \frac{p e^t}{1 - q e^t}.$$

$$\phi_X'(t) = \frac{p e^t (1 - q e^t) + p e^t \cdot q e^t}{(1 - q e^t)^2} = \frac{p e^t}{(1 - q e^t)^2}.$$

$$\phi_X'(0) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} = E(X).$$

$$\phi_X''(t) = \frac{p e^t (1 - q e^t)^2 + 2 p e^t (1 - q e^t) q e^t}{(1 - q e^t)^4}.$$

$$\phi_X''(0) = \frac{p^3 + 2 p^2 q}{p^4} = \frac{p + 2q}{p^2} = \frac{p + 2 - 2p}{p^2} = \frac{2-p}{p^2} = E(X^2)$$

$$Var(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}.$$

②
$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\phi_X(t) = E e^{tX} = \int_0^1 e^{tx} 1 \, dx = \frac{1}{t} e^{tx} \Big|_0^1 = \frac{e^t}{t} - \frac{1}{t}$$

$$= \frac{1}{t}(e^t - 1).$$

$$\phi_X'(t) = -\frac{1}{t^2}(e^t - 1) + \frac{1}{t} e^t \xrightarrow{t \to 0}$$

$$= \frac{1 - e^t + t e^t}{t^2}$$

$$\lim_{t \to 0} \phi_X'(t) = \lim_{t \to 0} \frac{-e^t + e^t + t e^t}{2t} = \frac{1}{2} = E(X).$$

$$\phi_X''(t) = \frac{2}{t^3}(e^t - 1) - \frac{1}{t^2} e^t - \frac{1}{t^2} e^t + \frac{1}{t} e^t.$$

$$= \frac{2(e^t - 1) - 2t e^t + t^2 e^t}{t^3} \xrightarrow{t \to 0} \frac{2e^t - 2e^t - 2te^t + 2te^t + t^2 e^t}{3t^2}$$

$$= \frac{1}{3} = E(X^2)$$

$$Var(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Example : let 2 independent continuous random variables X and Y with the same density function: $f_X = f_Y = f$.

Compute $P(X < Y) = ?$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

$$P(X < Y) = \iint_{(x<y)} f_X(x) f_Y(y)\, dx\, dy.$$

$$= \int_{\mathbb{R}} dy \left(\int_{-\infty}^{y} f_X(x)\, dx \right) f_Y(y)$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

$$= \int_{\mathbb{R}} F_X(y) f_Y(y)\, dy$$

$$= \int_{\mathbb{R}} dy \left(\int_{-\infty}^{y} f_X(x)\, dx \right) f_Y(y)$$

$$= \int_{\mathbb{R}} dy\; F_X(y) f_Y(y) = \int_{\mathbb{R}} F(y) f(y)\, dy.$$

$$f = F'$$

$$= \int_{\mathbb{R}} F(y) F'(y)\, dy = \frac{1}{2} F^2(y) \Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2} F^2(\infty) - \frac{1}{2} F^2(-\infty) = \frac{1}{2}.$$

Example: let X and Y two independent identically distributed (iid) random variables. Find the distribution of $X+Y$ in the following cases:

① $X, Y \sim Bin(p, n)$.

② $X, Y \sim Exp(\lambda)$.

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① $$\phi_{X+Y}(t) = E e^{t(X+Y)} = E(e^{tX} e^{tY}) = (Ee^{tX})(Ee^{tY})$$
$$= \phi_X(t)\,\phi_Y(t)$$
$$= (pe^t+1-p)^n (pe^t+1-p)^n$$
$$= (pe^t+1-p)^{2n}.$$

$X+Y \sim Bin(p, 2n)$.

② $$\phi_{X+Y}(t) = \phi_X(t)\,\phi_Y(t) = \frac{\lambda}{\lambda-t}\cdot\frac{\lambda}{\lambda-t} = \left(\frac{\lambda}{\lambda-t}\right)^2.$$

$X+Y \sim Gamma(\lambda, 1)$.

Conditional expectation:

$E(X|Y)$ is the conditional expectation of X given Y when $E(X|Y=y)$ is the conditional mean of X given $Y=y$.

* $$E(X|Y) = \begin{cases} \sum x f_{X|Y=y}(x) & \text{discrete} \\ \int x f_{X|Y=y}(x)\,dx & \text{Continuous} \end{cases}$$

* $$E(X) = E\,E(X|Y) = \begin{cases} \sum_y E(X|Y=y) f_Y(y). & \underline{\text{Discrete}} \\ \int E(X|Y=y) f_Y(y)\,dy & \underline{\text{Cont}} \end{cases}$$

Example: let Y /

y	-1	1	2
f_Y	$1/4$	$1/4$	$1/2$

Define:

$$X = \begin{cases} 1 & \text{if } Y \in \{-1, 2\}. \\ 0 & \text{if } Y = 1. \end{cases}$$

Compute $E(X)$?

$$E(X) = \sum_y E(X \mid Y=y)\, f_Y(y)$$

$$= E(X \mid Y=-1)\, f_Y(-1) + E(X \mid Y=1)\, f_Y(1) + E(X \mid Y=2)\, f_Y(2)$$

$$= 1 \cdot \tfrac{1}{4} + 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{2} = 3/4.$$

Example: let $Y \sim Exp(1)$.

$$X = \begin{cases} 1 & Y \geq 2 \\ -1 & Y \leq 2 \end{cases}.$$

$E(X) = ?$

$$E(X) = \mathbb{E}\,\mathbb{E}(X \mid Y) = \int E(X \mid Y=y)\, f_Y(y)\, dy.$$

$$= \int_0^\infty E(X \mid Y=y)\, e^{-y}\, dy$$

$$= \int_0^2 -e^{-y}\, dy + \int_2^\infty e^{-y}\, dy = e^{-y}\Big|_0^2 - e^{-y}\Big|_2^\infty$$

$$= e^{-2} - 1 - \left(0 - e^{-2}\right) = 2e^{-2} - 1.$$