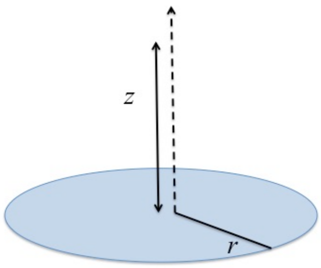
PHYS 507
1^{st} Midterm Exam – Spring 2020
Thursday 27^{th} February 2020

Instructor: Dr. V. Lempesis

Please answer all questions

1. Find the electric potential at a point on the axis of a disk and at a distance z away from the center of a disk with a surface charge density σ and radius r.



(5 marks)

Solution

The elementary charge dq shown in the figure creates an elementary potential dV at point A given by:

Vasileios Lembessis 1/3/2020 18:11
Comment [1]: Most of you tried to find the electric field at the point and the to find the potential. This is not wrong but it takes a lot of time and involves two integrations most of which you did the wrong.
Few of you found the electric field far away from the disk. And then found the potential. In this way you calculate the potential far away from the disk. But this is not what the problem asks for.

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{\rho^2+z^2}}$$

But $dq = \sigma dA = \sigma\rho d\rho d\phi$ so

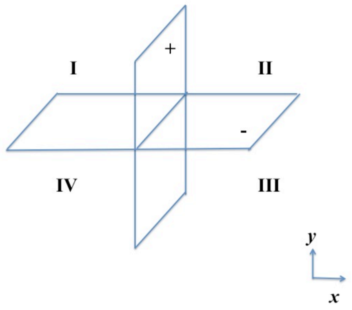
$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\rho d\rho d\phi}{\sqrt{\rho^2+z^2}} \Rightarrow V = \frac{\sigma}{4\pi\varepsilon_0}\int_0^R \frac{\rho}{\sqrt{\rho^2+z^2}}d\rho\int_0^{2\pi} d\phi \Rightarrow$$

$$V = \frac{\sigma 2\pi}{4\pi\varepsilon_0}\int_0^R \frac{\rho}{\sqrt{\rho^2+z^2}}d\rho \Rightarrow V = \frac{\sigma}{2\varepsilon_0}\frac{1}{2}\int_0^R \frac{1}{\sqrt{\rho^2+z^2}}d\rho^2 \Rightarrow$$

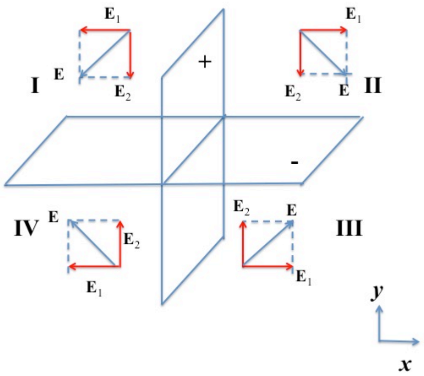
$$V = \frac{\sigma}{2\varepsilon_0}\frac{1}{2}\left.\frac{\left(\rho^2+z^2\right)^{1/2}}{1/2}\right|_0^R \Rightarrow V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-|z|\right]$$

2. Two infinite plane sheets with equal but opposite surface charge densities ($\pm\sigma$) divide the spece in four regions (I, II, III and IV) as shown in figure. Find the electric field (magnitude and direction) at each region. The axes directions are shown in the figure.

(5 marks)



Solution:



The total electric field is the resultant of the field $\mathbf{E}_1$ due to the positively charged sheet and of the field $\mathbf{E}_2$ of the negatively charged sheet. Both of them are everywhere perpendicular to each other and have the same magnitude $\sigma/2\varepsilon_0$. Thus for the total field we have:

Vasileios Lembessis 1/3/2020 18:07
Comment [2]: Most of you addded the two fields or subtracted like being scalars!!!! This is GREAT MISTAKE. The fields are VECTORS!!!

I. $\mathbf{E} = -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{x}} - \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{y}}\,,\ \ \theta = 225^0$

II. $\mathbf{E} = \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{x}} - \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{y}}\,,\ \ \theta = 315^0$

III. $\mathbf{E} = \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{x}} + \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{y}}\,,\ \ \theta = 45^0$

IV. $\mathbf{E} = -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{x}} + \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{y}}\,,\ \ \theta = 135^0$

$$E = \sqrt{E_1^2 + E_2^2} = \sqrt{\left(\frac{\sigma}{2\varepsilon_0}\right)^2 + \left(\frac{\sigma}{2\varepsilon_0}\right)^2} = \frac{\sigma\sqrt{2}}{2\varepsilon_0}$$

And the direction in each region is given in the figure.

3. A sphere of radius R carries a charge Q, which is uniformly distributed with a density in the volume of the sphere.
a) Find the electric field inside the sphere. (3 marks)
b) Find the electric field outside the sphere (3 marks).

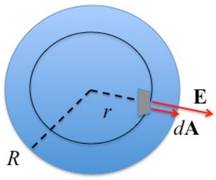
c) Use your answers at a) and b) and find the total energy of the configuration by using the formula $W = \frac{\varepsilon_0}{2}\int |\mathbf{E}|^2 d\tau$ (4 marks).

(10 marks)

Solution:

The distribution is characterized by a charge density $\rho = 3Q/(4\pi R^3)$.

a) Inside the sphere $(r < R)$. We consider a Gaussian surface of radius r as shown in figure. Due to radial symmetry the electric field $\mathbf{E}$ is parallel to the elementary surface area vector $d\mathbf{A}$. Gauss's law takes the form:

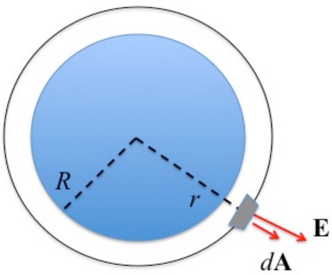


(a)

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0} \Rightarrow E\oint dA = \frac{\rho V}{\varepsilon_0} \Rightarrow E4\pi r^2 = \rho\frac{4}{3\varepsilon_0}\pi r^3$$

$$E = \frac{\rho r}{3\varepsilon_0} \Rightarrow E = \frac{3Qr}{4\cdot 3\pi\varepsilon_0 R^3} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 R^3} r$$

b) Outside the sphere $(r > R)$. We consider a Gaussian surface of radius r as shown in figure. Due to radial symmetry the electric field $\mathbf{E}$ is parallel to the elementary surface area vector $d\mathbf{A}$. Gauss's law takes the form:



(b)

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0} \Rightarrow E\oint dA = \frac{Q}{\varepsilon_0} \Rightarrow E4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

c) The energy is given by the formula

Vasileios Lembessis 1/3/2020 18:13
Comment [3]: Some of you considered also an integration term E_1E_2. This is not correct because the two fields do not overlap in a region of space. One is only inside the other is only outside.

$$W = \frac{\varepsilon_0}{2}\int |\mathbf{E}|^2\, d\tau = \frac{\varepsilon_0}{2}\left\{ \int\limits_{\substack{\text{inside}\\ \text{the sphere}}} |\mathbf{E}|^2\, d\tau + \int\limits_{\substack{\text{outside}\\ \text{the sphere}}} |\mathbf{E}|^2\, d\tau \right\}$$

$$W = \frac{\varepsilon_0}{2}\left\{ \left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)^2 \int_{r=0}^{R} r^4\, dr 2\times 2\pi + \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \int_{r=R}^{\infty} \frac{1}{r^2}\, dr 2\times 2\pi \right\}$$

$$W = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2 4\pi \left\{ \left(\frac{1}{R^3}\right)^2 \int_{r=0}^{R} r^4\, dr + \int_{r=R}^{\infty} \frac{1}{r^2}\, dr \right\}$$

$$W = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2 4\pi \left\{ \left(\frac{1}{R^3}\right)^2 \frac{R^5}{5} + \frac{1}{R} \right\} = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2 4\pi \frac{6}{5R} = \frac{Q^2}{4\pi\varepsilon_0}\frac{6}{10R} = \frac{3Q^2}{20\pi\varepsilon_0 R}$$

MATHEMATICAL SUPPLEMENT

- CYLINDRICAL COORDINATES

  Surface element on x-y plane: $dA = \rho d\rho d\varphi$

  $0 \le \varphi \le 2\pi$

- SPHERICAL COORDINATES

  Elementary volume: $d\tau = r^2 dr \sin\theta d\theta d\varphi$

  Elementary surface on a sphere of radius *r*: $dA = r^2 \sin\theta d\theta d\varphi$

  $0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$