

**PHYS 507**  
**1<sup>st</sup> Midterm Exam – Spring 2021 – Solutions**  
**Wednesday 3<sup>rd</sup> March 2021**

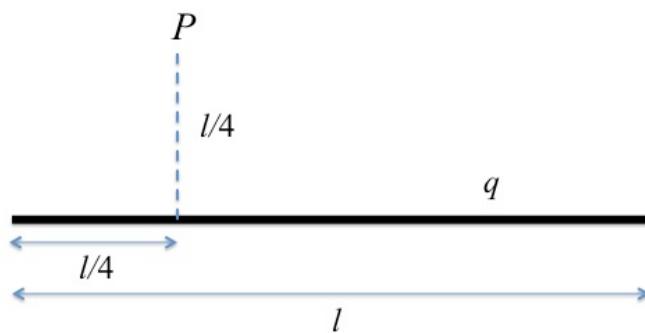
**Instructor: Professor V. Lempesis**

*Please answer all questions*

1. The rod in the figure has a positive charge  $q$  and a length  $l$ . The charge is uniformly distributed along the rod. Find the electric field at the point P.

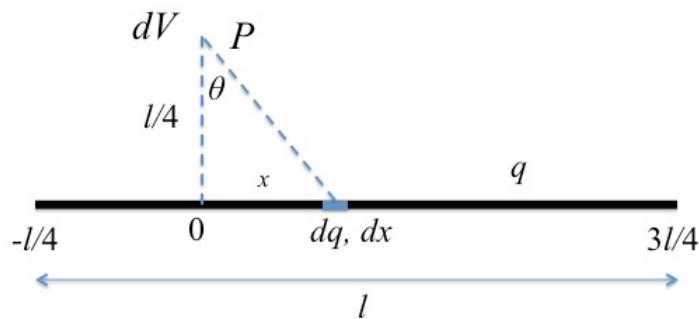
You are given:  $\int \frac{dx}{(x^2 + k)^{1/2}} = \ln\left(x + \sqrt{x^2 + k}\right)$

(5 marks)



**Solution**

The rod has a linear charge density  $\lambda = q/l$



The elementary charge  $dq$  shown in the figure creates an elementary potential at P

$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + l^2/16)^{1/2}} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{(x^2 + l^2/16)^{1/2}}$$

To find the total potential we need to integrate from  $x = -l/4$  to  $x = 3l/4$ .

Thus

$$\begin{aligned}
V &= \int_{x=-l/4}^{x=3l/4} dV = \frac{\lambda}{4\pi\epsilon_0} \int_{x=-l/4}^{x=3l/4} \frac{dx}{(x^2 + l^2/16)^{1/2}} = \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[ \ln \left( x + \sqrt{x^2 + l^2/16} \right) \right]_{x=-l/4}^{x=3l/4} = \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[ \ln \left( 3l/4 + \sqrt{9l^2/16 + l^2/16} \right) - \ln \left( -l/4 + \sqrt{l^2/16 + l^2/16} \right) \right] = \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[ \ln \left( 3l/4 + \sqrt{10l^2/16} \right) - \ln \left( -l/4 + \sqrt{l^2/8} \right) \right] = \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[ \ln \left( \frac{3l/4 + \sqrt{10l^2/16}}{-l/4 + \sqrt{l^2/8}} \right) \right] = \frac{q}{4\pi\epsilon_0 l} \left[ \ln \left( \frac{3l/4 + \sqrt{10l^2/16}}{-l/4 + \sqrt{l^2/8}} \right) \right]
\end{aligned}$$

2. A distribution of electric charge lies within an infinite length cylinder of radius  $a$ . In cylindrical coordinates the charge density at a radial distance  $r$  ( $r < a$ ) is given by:  $\rho = C(a - r)/\pi a^3$ . Calculate, (I) the electric field and (II) the potential in all space. Assume that the potential is zero at the surface of the cylinder.

(8 marks)

**Solution:**

I) Due to the radial dependence of the density the electric field will have only radial component. There are two regions:

- a) Inside the cylinder  $0 < r < a$ .

We choose as our Gaussian surface a cylinder of length  $l$  (along the z-direction) and of radius  $\rho < a$ .

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{encl}}{\epsilon_0} \Rightarrow E \oint dA = \frac{1}{\epsilon_0} \int_{cylinder} \rho dV \Rightarrow$$

$$E \oint dA = \frac{1}{\epsilon_0} \int_{cylinder} \rho r dr d\phi dz \Rightarrow$$

$$E 2\pi r l = \frac{C}{\epsilon_0 \pi a^3} \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^l dz \int_{r'=0}^r r'(a - r') dr' \Rightarrow$$

$$E 2\pi r l = \frac{C}{\epsilon_0 \pi a^3} 2\pi l \left( -\frac{1}{3} r^3 + \frac{1}{2} a r^2 \right) \Rightarrow$$

$$E = \frac{Cr}{\epsilon_0 2\pi a^3} \left( a - \frac{2}{3} r \right)$$

- b) When we are outside the cylinder then we choose as our Gaussian surface a cylinder of length  $l$  (along the z-direction) and of radius  $\rho > a$

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{encl}}{\epsilon_0} \Rightarrow E \oint dA = \frac{1}{\epsilon_0} \int_{cylinder} \rho dV \Rightarrow$$

$$E \oint dA = \frac{1}{\epsilon_0} \int_{cylinder} \rho r dr d\phi dz \Rightarrow$$

$$E 2\pi r l = \frac{C}{\epsilon_0 \pi a^3} \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^l dz \int_{r=0}^a r(a-r) dr \Rightarrow$$

$$E 2\pi r l = \frac{C}{\epsilon_0 \pi a^3} 2\pi l \left( -\frac{1}{3} a^3 + \frac{1}{2} a a^2 \right) \Rightarrow$$

$$E = \frac{C}{6\epsilon_0 \pi r}$$

II) Since we work in cylindrical coordinates and the electric field has only radial components then  $E = -\partial V / \partial r$  thus in order to find the potential we need to integrate this formula. The potential is given in the two regions as:

- a) For  $0 < r < a$  we have

$$V(\rho) = - \int_{\rho}^r \mathbf{E} \cdot d\mathbf{l} = - \int_{\rho}^r E dr' = - \frac{C}{\epsilon_0 2\pi a^3} \int_a^r r' \left( a - \frac{2}{3} r' \right) dr' = - \frac{C}{\epsilon_0 2\pi a^3} \left[ \frac{2}{9} (a^3 - r^3) + \frac{1}{2} a (r^2 - a^2) \right]$$

- b) For  $\rho > a$  we have

$$V(\rho) = - \int_{\rho}^r \mathbf{E} \cdot d\mathbf{l} = - \int_{\rho}^r E dr' = - \frac{C}{6\epsilon_0 \pi} \int_a^r \frac{1}{r'} dr' = - \frac{C}{6\epsilon_0 \pi} [\ln r - \ln a] = - \frac{C}{6\epsilon_0 \pi} \ln \left( \frac{r}{a} \right)$$

3. Assume that the electrostatic potential (spherical coordinates) is given by

$V(r) = A \frac{e^{-\lambda r}}{r}$ , where  $A$  and  $\lambda$  are positive constants. Find (i) the electric field, (ii) the charge density and (iii) the total charge. You are given that:

$$\nabla \cdot \left( \frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r}), \quad \int_{\text{all space}} f(\mathbf{r}) \delta^3(\mathbf{r} - \mathbf{a}) dV = f(\mathbf{a}), \quad f(x)\delta(x) = f(0)\delta(x),$$

$$\int_{r=0}^{\infty} e^{-\lambda r} r dr = \frac{1}{\lambda^2}.$$

(5 marks)

**Solution:**

(i)

$$\mathbf{E} = -\nabla V = -A \nabla \left( \frac{e^{-\lambda r}}{r} \right) = -A \frac{\partial}{\partial r} \left( \frac{e^{-\lambda r}}{r} \right) \hat{\mathbf{r}} =$$

$$-A \left\{ \frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2} \right\} \hat{\mathbf{r}} = Ae^{-\lambda r} (1 + \lambda r) \frac{\hat{\mathbf{r}}}{r^2}$$

(ii)

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 \nabla \cdot \left\{ Ae^{-\lambda r} (1 + \lambda r) \frac{\hat{\mathbf{r}}}{r^2} \right\} =$$

$$\epsilon_0 A \left\{ e^{-\lambda r} (1 + \lambda r) \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} + \frac{\hat{\mathbf{r}}}{r^2} \nabla e^{-\lambda r} (1 + \lambda r) \right\} =$$

$$\epsilon_0 A \left\{ e^{-\lambda r} (1 + \lambda r) 4\pi \delta^3(\mathbf{r}) - \left( \frac{\lambda^2}{r} e^{-\lambda r} \right) \right\}_{f(x)\delta(x)=f(0)\delta(x)} =$$

$$\epsilon_0 A \left\{ 4\pi \delta^3(\mathbf{r}) - \left( \frac{\lambda^2}{r} e^{-\lambda r} \right) \right\}$$

(iii)

$$Q = \int \rho d\tau = \epsilon_0 A \int \left\{ 4\pi \delta^3(\mathbf{r}) - \left( \frac{\lambda^2}{r} e^{-\lambda r} \right) \right\} d\tau =$$

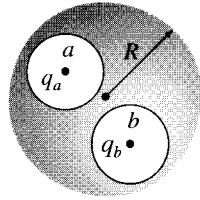
$$\epsilon_0 A \left\{ \int 4\pi \delta^3(\mathbf{r}) d\tau - \lambda^2 \int \frac{1}{r} e^{-\lambda r} d\tau \right\} =$$

$$\epsilon_0 A \left\{ 4\pi - \lambda^2 \int_{r=0}^{\infty} \frac{1}{r} e^{-\lambda r} r^2 dr \int_{\theta=0}^{\pi} \sin \theta d\theta \int_{\phi=0}^{2\pi} d\phi \right\} =$$

$$\epsilon_0 A \left\{ 4\pi - 4\pi \lambda^2 \int_{r=0}^{\infty} e^{-\lambda r} r dr \right\} =$$

$$4\pi \epsilon_0 A \left\{ 1 - \lambda^2 \frac{1}{\lambda^2} \right\} = 0$$

4. Two spherical cavities, of radii  $a$  and  $b = 2a$ , are hollowed out from the interior of a (neutral) conducting sphere of radius  $R$ . At the center of each cavity a point charge is placed – call these charges  $q_a = 2q$  and  $q_b = -q$  ( $q > 0$ ). (a) Find the surface charge densities  $\sigma_a$ ,  $\sigma_b$  and  $\sigma_R$ . (b) What is the field outside the conductor? (c) What is the field within each cavity? (d) What is the net force on  $q_a$ . (2 marks)



**Solution:**

$$(a) \sigma_a = -\frac{q_a}{4\pi a^2} = -\frac{2q}{4\pi a^2}, \quad \sigma_b = -\frac{q_b}{4\pi b^2} = -\frac{-q}{4\pi (2a)^2} = \frac{q}{16\pi a^2}$$

$$(b) \mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{(q_a + q_b)}{R^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{(2q - q)}{R^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{\mathbf{r}}$$

$$(c) \mathbf{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{2q}{r^2} \hat{\mathbf{r}} = \frac{1}{2\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}, \\ \mathbf{E}_b = \frac{1}{4\pi\epsilon_0} \frac{q_b}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{-q}{r^2} \hat{\mathbf{r}} = -\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

(d) Zero

## MATHEMATICAL SUPPLEMENT

- CYLINDRICAL COORDINATES

Surface element on x-y plane:  $dA = \rho d\rho d\varphi$

Elementary volume:  $d\tau = \rho d\rho d\varphi dz$

$$0 \leq \varphi \leq 2\pi$$

Unit vectors  $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}\}$ ,  $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{z}}$ ,  $\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_z \hat{\mathbf{z}}$ .

$$\begin{aligned} x &= r \cos \theta & r &= \sqrt{x^2 + y^2} \\ y &= r \sin \theta & \tan \theta &= \frac{y}{x} \\ z &= z & z &= z \end{aligned}$$

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{\partial \psi}{\partial z} \hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left( \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{\mathbf{r}} + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left( \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{\mathbf{z}}$$

Note that

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{r}} & r \hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & r A_\theta & A_z \end{vmatrix}.$$

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

- SPHERICAL COORDINATES

Elementary volume:  $d\tau = r^2 dr \sin \theta d\theta d\varphi$

Elementary surface on a sphere of radius  $r$ :  $dA = r^2 \sin \theta d\theta d\varphi$

$$0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi$$

Unit vectors  $\{\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}\}$ ,  $\hat{\mathbf{r}} \times \hat{\theta} = \hat{\phi}$ ,  $\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$ .

$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \cos \theta &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ z &= r \cos \theta & \tan \phi &= \frac{y}{x} \end{aligned}$$

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi$$

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \hat{\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\begin{aligned} \nabla \times \mathbf{A} &= \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ &+ \left[ \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi} \end{aligned}$$

Note that

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{r}} & r \hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin \theta A_\phi \end{vmatrix}.$$

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}$$