PHYS 505
1st Midterm Exam
Monday 30th October 2017

Instructor: Dr. V. Lempesis

Student Grade: ……./20

Please answer all questions

1. The electron in a hydrogen atom is in a state given by:

$$\psi = A\psi_{100} + 2A\psi_{21-1}$$

(a) Find A.
(b) Find $\langle \mathbf{l}^2 \rangle$, $\langle l_z \rangle$, $\langle E \rangle$, Δl_z the average energy of the electron
(c) Find the time evolution of the state ψ of the atom.

(6 marks)

Solution:
(a) The wave function must be normalized thus

$$\langle \psi | \psi \rangle = 1 \Rightarrow \left(A^* \langle \psi_{100} | + 2A^* \langle \psi_{21-1} | \right) \cdot \left(A | \psi_{100} \rangle + 2A | \psi_{21-1} \rangle \right) = |A|^2 \langle \psi_{100} | \psi_{100} \rangle + 2|A|^2 \langle \psi_{100} | \psi_{21-1} \rangle +$$
$$2|A|^2 \langle \psi_{21-1} | \psi_{100} \rangle + 4|A|^2 \langle \psi_{21-1} | \psi_{21-1} \rangle = 1 \Rightarrow 5|A|^2 = 1 \Rightarrow A = \frac{1}{\sqrt{5}}$$

Thus the probabilities with which the two states appear are:

$$P_{100} = A^2 = \frac{1}{5} \text{ and } P_{21-1} = (2A)^2 = \frac{4}{5}.$$

(b) To find the average values of the relevant quantities we construct the following table:

state	l^2	l_z	E
$\lvert\psi_{100}\rangle$	0	0	E_1
$\lvert\psi_{21-1}\rangle$	$2\hbar^2$	$-\hbar$	E_2

Thus we have:

i) $\left\langle l^2 \right\rangle = 0 \cdot P_{100} + 2\hbar^2 P_{21-1} = \frac{8}{5}\hbar^2$.

ii) $\left\langle l_z \right\rangle = 0 \cdot P_{100} - \hbar P_{21-1} = -\frac{4}{5}\hbar$

iii)

$$\left\langle E \right\rangle = E_1 \cdot P_{100} + E_2 P_{21-1} = E_1 \cdot P_{100} + \frac{E_1}{4} P_{21-1} = E_1 \left(P_{100} + \frac{1}{4} P_{21-1} \right) =$$

$$E_1 \left(\frac{1}{5} + \frac{1}{4}\frac{4}{5} \right) = \frac{2}{5} E_1 = -13.6 \frac{2}{5} eV = -5.44 eV$$

iv) To find Δl_z we need to calculate $\left\langle l_z^2 \right\rangle$.

$$\left\langle l_z^2 \right\rangle = 0 \cdot P_{100} + (-\hbar)^2 P_{21-1} = \frac{4}{5}\hbar^2 .$$

$$\Delta l_z = \sqrt{\left\langle l_z^2 \right\rangle - \left\langle l_z \right\rangle^2} = \sqrt{\frac{4}{5}\hbar^2 - \left(-\frac{4}{5}\hbar \right)^2} = \sqrt{\frac{4}{5}\hbar^2 - \frac{16}{25}\hbar^2} = \hbar\sqrt{\frac{4}{5} - \frac{16}{25}} = \hbar\sqrt{\frac{4}{25}} = \frac{2}{5}\hbar$$

2. An electron is in the spin state:

$$\chi = A \begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$$

(a) Determine the constant A by normalizing χ.

(2 marks)

(b) If you measured s_z on this electron, what values could you get, and what is the probability of each? What is the expectation value (average value) of s_z?

(6 marks)

(a) The spin state is normalized thus:

$$\left\langle \chi | \chi \right\rangle = 1 \Rightarrow A^* \begin{pmatrix} 1+2i & 2 \end{pmatrix} A \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = 1 \Rightarrow |A|^2 \left[(1+2i) \cdot (1-2i) + 2 \cdot 2 \right] = 1 \Rightarrow$$

$$|A|^2 (1+4+4) = 1 \Rightarrow |A|^2 = 1/9 \Rightarrow A = 1/3$$

(b) The values of the spin will be $s_z = \pm\hbar/2$. The relevant probabilities are given by:

$$P_{1/2} = \left|\langle\uparrow|x\rangle\right|^2 = \left|\frac{1}{3}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}\right|^2 = \frac{1}{9}\left|1-2i\right|^2 = \frac{1}{9}(1+4) = \frac{5}{9}$$

Thus

$$P_{1/2} + P_{-1/2} = 1 \Rightarrow P_{-1/2} = 1 - P_{1/2} \Rightarrow P_{-1/2} = 1 - \frac{5}{9} = \frac{4}{9}$$

The average value of the spin in the z-direction is

$$\langle s_z \rangle = \left(\frac{\hbar}{2}\right)P_{1/2} + \left(-\frac{\hbar}{2}\right)P_{-1/2} = \left(\frac{\hbar}{2}\right)\frac{5}{9} - \left(\frac{\hbar}{2}\right)\frac{4}{9} = \left(\frac{1}{9}\cdot\frac{\hbar}{2}\right) = \frac{\hbar}{18}.$$

3. Find the eigenvalues and eigenfunctions of the operators:

a) $l_x^2 + l_y^2$

b) $l_x^2 + l_y^2 - l_z^4$

c) $l_- l_+^2 l_-$

Solutions:

a) $l_x^2 + l_y^2 + l_z^2 = l^2 \Rightarrow l_x^2 + l_y^2 = l^2 - l_z^2$ Thus this operator has the same eigenfunctions as l^2, l_z i.e. the spherical harmonics. So the average value is given by:

$$\left\langle Y_l^m \middle| \left(l_x^2 + l_y^2\right) \middle| Y_l^m \right\rangle = \left\langle Y_l^m \middle| \left(l^2 - l_z^2\right) \middle| Y_l^m \right\rangle = \left\langle Y_l^m \middle| l^2 \middle| Y_l^m \right\rangle - \left\langle Y_l^m \middle| l_z^2 \middle| Y_l^m \right\rangle = \hbar^2 l(l+1) - m^2\hbar^2 =$$
$$\hbar^2\left[l(l+1) - m^2\right]$$

b) Similarly we have:

$$l_x^2 + l_y^2 - l_z^4 = l^2 - l_z^2 - l_z^4.$$

Thus the operator $l_x^2 + l_y^2 - l_z^4$ has the same eigenfunctions as l^2, l_z i.e. the spherical harmonics. So the average value is given by:

$$\left\langle Y_l^m \left| \left(l_x^2 + l_y^2 - l_z^4\right) \right| Y_l^m \right\rangle = \left\langle Y_l^m \left| \left(l^2 - l_z^2 - l_z^4\right) \right| Y_l^m \right\rangle = \left\langle Y_l^m \left| l^2 \right| Y_l^m \right\rangle - \left\langle Y_l^m \left| l_z^2 \right| Y_l^m \right\rangle - \left\langle Y_l^m \left| l_z^4 \right| Y_l^m \right\rangle$$
$$= \hbar^2 l\left(l+1\right) - m^2\hbar^2 - m^4\hbar^4$$

c) The operator $l_- l_+^2 l_-$ contains the same number of raising and lowering operators, so when it acts on a spherical harmonic it will return us to the same harmonic. Thus the spherical harmonics are eigenstates of this operator as well.

$$l_- l_+^2 l_- = l_- l_+ l_+ l_- = \left(l_- l_+\right) \cdot \left(l_+ l_-\right)$$

But

$$l_- l_+ = \left(l_x - il_y\right) \cdot \left(l_x + il_y\right) = l_x^2 + l_y^2 - il_x l_y + il_y l_x = l^2 - l_z^2 - i\left(l_x l_y - l_y l_x\right) =$$
$$l^2 - l_z^2 - i\left[l_x, l_y\right] = l^2 - l_z^2 - \hbar l_z$$

similarly

$$l_+ l_- = l^2 - l_z^2 + \hbar l_z \, .$$

Thus

$$\left\langle Y_l^m \left| l_- l_+^2 l_- \right| Y_l^m \right\rangle = \left\langle Y_l^m \left| \left(l^2 - l_z^2 - \hbar l_z\right) \cdot \left(l^2 - l_z^2 + \hbar l_z\right) \right| Y_l^m \right\rangle =$$
$$\left[l\left(l+1\right) - m^2 - m\right]\hbar^2 \cdot \left[l\left(l+1\right) - m^2 + m\right]\hbar^2 =$$
$$\left\{\left[l\left(l+1\right) - m^2\right]^2 - m^2\right\}\hbar^4$$

Attention: in the last expression we used the hermiticity property of the operators thus

$$\left\langle Y_l^m \right| \left(l^2 - l_z^2 - \hbar l_z\right) = \left(\left(l^2 - l_z^2 - \hbar l_z\right) \left| Y_l^m \right\rangle\right)^+ = \left\{\left[l\left(l+1\right) - m^2 - m\right]\hbar^2 \left| Y_l^m \right\rangle\right\}^+ =$$
$$\left\langle Y_l^m \right| \left[l\left(l+1\right) - m^2 - m\right]\hbar^2$$

Mathematical Supplement:

- For any physical quantity A: $\Delta A = \sqrt{\left\langle A^2 \right\rangle - \left\langle A \right\rangle^2}$
- $\left[x, p_x\right] = i\hbar, \ \left[y, p_y\right] = i\hbar, \ \left[z, p_z\right] = i\hbar$
- For any type of angular momentum: $k = l, s$.

$$k_+\left|k,m_k\right\rangle = \hbar\sqrt{k\left(k+1\right)-m_k\left(m_k+1\right)}\left|k,m_k+1\right\rangle$$
$$k_-\left|k,m_k\right\rangle = \hbar\sqrt{k\left(k+1\right)-m_k\left(m_k-1\right)}\left|k,m_k-1\right\rangle$$

- $l_\pm = l_x \pm i l_y$
- $l_x = i\hbar\left\{\sin\varphi\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right\}, \quad l_y = i\hbar\left\{-\cos\varphi\frac{\partial}{\partial\theta}+\frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right\}$
- $\left[l_z, l_x\right]=i\hbar l_y,\ \left[l_x, l_y\right]=i\hbar l_z,\ \left[l_y, l_z\right]=i\hbar l_x$
- [A, BC]=B[A,C]+[A, B]C

- For a particle with spin 1/2

$$s_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad s_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad s_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\underbrace{X_\uparrow = \begin{pmatrix}1\\0\end{pmatrix}}_{\text{spin up}},\quad \underbrace{X_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}}_{\text{spin down}}$$

- For a particle with spin s=1

$$\underbrace{X_\uparrow = \begin{pmatrix}1\\0\\0\end{pmatrix}}_{\text{spin up}},\quad \underbrace{X_- = \begin{pmatrix}0\\1\\0\end{pmatrix}}_{\text{spin horizontal}},\quad \underbrace{X_\downarrow = \begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{spin down}}$$

$$s_y = \frac{\sqrt{2}}{2}\hbar\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$$