Lecture 5

Euclidean n-space R^n and Dot, Cross Products

5.0 Vectors in R^n

5.1 Length of a vector

5.2 Dot Product

5.4 Cross Product

5.0 Vectors in R^n

- An ordered *n*-tuple:

  a sequence of *n* real number $(x_1, x_2, \cdots, x_n)$

- *n*-space: R^n

  the set of all ordered n-tuple

- Ex:

$n = 1$ R^1 = 1-space

= set of all real number

$n = 2$ R^2 = 2-space

= set of all ordered pair of real numbers (x_1, x_2)

$n = 3$ R^3 = 3-space

= set of all ordered triple of real numbers (x_1, x_2, x_3)

$n = 4$ R^4 = 4-space

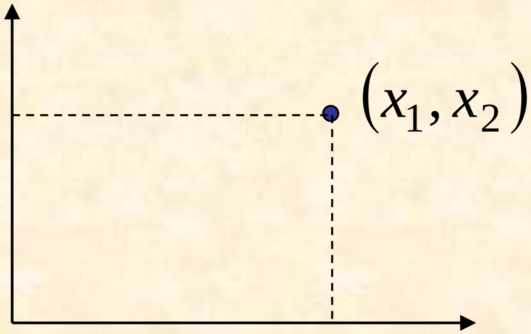
= set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

- Notes:

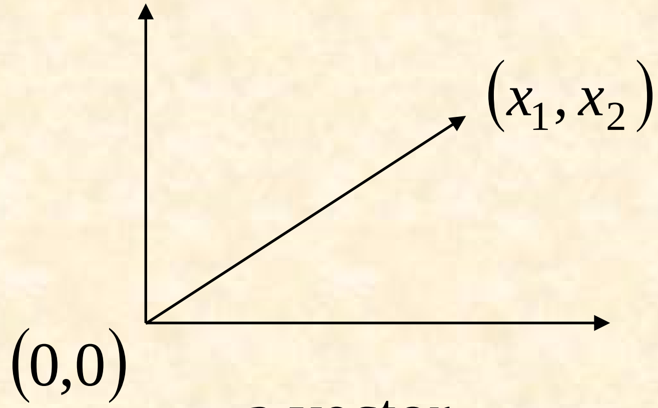
  (1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a point</u> in R^n with the x_i's as its coordinates.

  (2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a vector</u> $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.

- Ex:



a point



a vector

$\mathbf{u} = (u_1, u_2, \cdots, u_n),\ \mathbf{v} = (v_1, v_2, \cdots, v_n)$ (two vectors in R^n)

- Equal:

  $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1,\ u_2 = v_2, \cdots, u_n = v_n$

- Vector addition (the sum of **u** and **v**):

  $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$

- Scalar multiplication (the scalar multiple of **u** by *c*):

  $c\mathbf{u} = (cu_1, cu_2, \cdots, cu_n)$

- Notes:

  The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n.

- Negative:

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$$

- Difference:

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n)$$

- Zero vector:

$$\mathbf{0} = (0, 0, \ldots, 0)$$

- Notes:

(1) The zero vector $\mathbf{0}$ in R^n is called the **additive identity** in R^n.

(2) The vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$.

- Thm 4.2: (Properties of vector addition and scalar multiplication)

Let **u**, **v**, and **w** be vectors in R^n , and let c and d be scalars.

(1) **u**+**v** is a vector in R^n

(2) **u**+**v** = **v**+**u**

(3) (**u**+**v**)+**w** = **u**+(**v**+**w**)

(4) **u**+**0** = **u**

(5) **u**+(–**u**) = **0**

(6) c**u** is a vector in R^n

(7) c(**u**+**v**) = c**u**+c**v**

(8) (c+d)**u** = c**u**+d**u**

(9) c(d**u**) = (cd)**u**

(10) 1(**u**) = **u**

- Ex 5: (Vector operations in R^4)

Let **u**=(2, – 1, 5, 0), **v**=(4, 3, 1, – 1), and **w**=(– 6, 2, 0, 3) be vectors in R^4. Solve **x** for x in each of the following.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u} - \mathbf{v}+\mathbf{x}$

Sol: (a)
$$\begin{aligned}\mathbf{x} &= 2\mathbf{u}-(\mathbf{v}+3\mathbf{w})\\ &= 2\mathbf{u}-\mathbf{v}-3\mathbf{w}\\ &= (4,-2,10,0)-(4,3,1,-1)-(-18,6,0,9)\\ &= (4-4+18,-2-3-6,10-1-0,0+1-9)\\ &= (18,-11,9,-8).\end{aligned}$$

$$\text{(b)}\quad 3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u}-\mathbf{v}+\mathbf{x}$$

$$3\mathbf{x}+3\mathbf{w} = 2\mathbf{u}-\mathbf{v}+\mathbf{x}$$

$$3\mathbf{x}-\mathbf{x} = 2\mathbf{u}-\mathbf{v}-3\mathbf{w}$$

$$2\mathbf{x} = 2\mathbf{u}-\mathbf{v}-3\mathbf{w}$$

$$\mathbf{x} = \mathbf{u}-\tfrac{1}{2}\mathbf{v}-\tfrac{3}{2}\mathbf{w}$$

$$= (2,1,5,0)+\left(-2,\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{1}{2}\right)+\left(9,-3,0,\tfrac{-9}{2}\right)$$

$$= \left(9,\tfrac{-11}{2},\tfrac{9}{2},-4\right)$$

- Thm 4.3: (Properties of additive identity and additive inverse)

Let **v** be a vector in R^n and c be a scalar. Then the following is true.

(1) The additive identity is unique. That is, if $\mathbf{u}+\mathbf{v}=\mathbf{v}$, then $\mathbf{u} = \mathbf{0}$

(2) The additive inverse of **v** is unique. That is, if $\mathbf{v}+\mathbf{u}=\mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$

(3) $0\mathbf{v}=\mathbf{0}$

(4) $c\mathbf{0}=\mathbf{0}$

(5) If $c\mathbf{v}=\mathbf{0}$, then $c=0$ or $\mathbf{v}=\mathbf{0}$

(6) $-(-\mathbf{v}) = \mathbf{v}$

- Linear combination:

The vector **x** is called a **linear combination** of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v}_n$,

if it can be expressed in the form

$$\mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v}_n \qquad c_1, c_2, \cdots, c_n : \text{scalar}$$

- Ex 6:

Given **x** = (– 1, – 2, – 2), **u** = (0,1,4), **v** = (– 1,1,2), and **w** = (3,1,2) in R^3, find *a*, *b*, and *c* such that **x** = *a***u**+*b***v**+*c***w**.

Sol:

$$\begin{aligned} -b + 3c &= -1 \\ a + b + c &= -2 \\ 4a + 2b + 2c &= -2 \end{aligned}$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1$$

Thus $\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$

- Notes:

A vector $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ in R^n can be viewed as:

a $1\times n$ row matrix (row vector): $\mathbf{u}=[u_1,\ u_2,\cdots,u_n]$

or

a $n\times 1$ column matrix (column vector): $\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

(The matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations)

Vector addition

$$\mathbf{u}+\mathbf{v}=(u_1, u_2, \cdots, u_n)+(v_1, v_2, \cdots, v_n)$$
$$=(u_1+v_1, u_2+v_2, \cdots, u_n+v_n)$$

$$\mathbf{u}+\mathbf{v}=[u_1, u_2, \cdots, u_n]+[v_1, v_2, \cdots, v_n]$$
$$=[u_1+v_1, u_2+v_2, \cdots, u_n+v_n]$$

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}+\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}$$

Scalar multiplication

$$c\mathbf{u}=c(u_1,\ u_2, \cdots,\ u_n)$$
$$=(cu_1, cu_2, \cdots,\ cu_n)$$

$$c\mathbf{u}=c[u_1,\ u_2, \cdots,\ u_n]$$
$$=[cu_1, cu_2, \cdots,\ cu_n]$$

$$c\mathbf{u}=c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}=\begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

5.1 Length and Dot Product in R^n

- Length:

The length of a vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ in R^n is given by

$$\|\mathbf{v}\| = \sqrt{v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2}$$

- Notes: The length of a vector is also called its **norm**.

- Notes: Properties of length

(1) $\|\mathbf{v}\| \geq 0$

(2) $\|\mathbf{v}\| = 1 \Rightarrow \mathbf{v}$ is called a **unit vector**.

(3) $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = 0$

(4) $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

- Ex 1:

(a) In R^5, the length of $\mathbf{v}=(0\,,-2\,,1\,,4\,,-2)$ is given by

$$\|\mathbf{v}\|=\sqrt{0^2+(-2)^2+1^2+4^2+(-2)^2}=\sqrt{25}=5$$

(b) In R^3 the length of $\mathbf{v}=(\frac{2}{\sqrt{17}}\,,\frac{-2}{\sqrt{17}}\,,\frac{3}{\sqrt{17}})$ is given by

$$\|\mathbf{v}\|=\sqrt{\left(\frac{2}{\sqrt{17}}\right)^2+\left(\frac{-2}{\sqrt{17}}\right)^2+\left(\frac{3}{\sqrt{17}}\right)^2}=\sqrt{\frac{17}{17}}=1$$

($\mathbf{v}$ is a unit vector)

- A standard unit vector in R^n:

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} = \{(1,0,\cdots,0), (0,1,\cdots,0), (0,0,\cdots,1)\}$$

- Ex:

the standard unit vector in R^2: $\{i, j\} = \{(1,0), (0,1)\}$

the standard unit vector in R^3: $\{i, j, k\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

- Notes: (Two nonzero vectors are parallel)

$$\mathbf{u} = c\mathbf{v}$$

(1) $c > 0 \Rightarrow$ **u** and **v** have the same direction

(2) $c < 0 \Rightarrow$ **u** and **v** have the opposite direction

- Thm 5.1: (Length of a scalar multiple)

Let **v** be a vector in R^n and c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

Pf:

$$\mathbf{v} = (v_1\,, v_2\,, \cdots, v_n)$$

$$\Rightarrow c\mathbf{v} = (cv_1\,, cv_2\,, \cdots, cv_n)$$

$$\|c\mathbf{v}\| = \|\,(cv_1\,, cv_2\,, \cdots, cv_n)\,\|$$

$$= \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}$$

$$= \sqrt{c^2(v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2)}$$

$$= |c|\sqrt{v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2}$$

$$= |c|\,\|\mathbf{v}\|$$

- Thm 5.2: (Unit vector in the direction of **v**)

  If **v** is a nonzero vector in R^n, then the vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

  has length 1 and has the same direction as **v**. This vector **u** is called the **unit vector in the direction of v**.

Pf:

$$\mathbf{v} \text{ is nonzero } \Rightarrow \|\mathbf{v}\| \neq 0 \Rightarrow \frac{1}{\|\mathbf{v}\|} > 0$$

$$\Rightarrow \mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \quad (\mathbf{u} \text{ has the same direction as } \mathbf{v})$$

$$\|\mathbf{u}\| = \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1 \quad (\text{u has length 1})$$

- Notes:

(1) The vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is called the unit vector in the direction of **v**.

(2) The process of finding the unit vector in the direction of **v** is called **normalizing** the vector **v.**

- Ex 2: (Finding a unit vector)

Find the unit vector in the direction of $\mathbf{v}=(3,-1,2)$,

and verify that this vector has length 1.

Sol:

$$\mathbf{v}=(3,-1,2) \quad \Rightarrow \|\mathbf{v}\|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}$$

$$\Rightarrow \frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{(3,-1,2)}{\sqrt{3^2+(-1)^2+2^2}}=\frac{1}{\sqrt{14}}(3,-1,2)=\left(\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}},\frac{2}{\sqrt{14}}\right)$$

$$\because \quad \sqrt{\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-1}{\sqrt{14}}\right)^2+\left(\frac{2}{\sqrt{14}}\right)^2}=\sqrt{\frac{14}{14}}=1$$

$\therefore \quad \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

- Distance between two vectors:

The **distance** between two vectors **u** and **v** in R^n is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- Notes: (Properties of distance)

(1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Ex 3: (Finding the distance between two vectors)

The distance between **u**=(0, 2, 2) and **v**=(2, 0, 1) is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(0-2, 2-0, 2-1)\|$$

$$= \sqrt{(-2)^2 + 2^2 + 1^2} = 3$$

5.2 Dot Product

- Dot product in R^n:

  The **dot product** of $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$

  is the scalar quantity

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Ex 4: (Finding the dot product of two vectors)

  The dot product of **u**=(1, 2, 0, -3) and **v**=(3, -2, 4, 2) is

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7$$

- Thm 5.3: (Properties of the dot product)

If **u**, **v**, and **w** are vectors in R^n and c is a scalar,

then the following properties are true.

(1) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

(2) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

(3) $c(\mathbf{u}\cdot\mathbf{v})=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})$

(4) $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

(5) $\mathbf{v}\cdot\mathbf{v}\geq 0$, and $\mathbf{v}\cdot\mathbf{v}=0$ if and only if

$\boldsymbol{v}=0$

- Euclidean *n*-space:

  R^n was defined to be the *set* of all order n-tuples of real numbers. When R^n is combined with the standard operations of vector addition, scalar multiplication, vector length, and the dot product, the resulting vector space is called **Euclidean *n*-space**.

- Ex 5: (Finding dot products)

$$\mathbf{u}=(2\,,-2)\,,\ \mathbf{v}=(5\,,8),\mathbf{w}=(-4\,,3)$$

(a) $\mathbf{u}\cdot\mathbf{v}$ (b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (c) $\mathbf{u}\cdot(2\mathbf{v})$ (d) $\|\mathbf{w}\|^2$ (e) $\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w})$

Sol:

(a) $\mathbf{u}\cdot\mathbf{v}=(2)(5)+(-2)(8)=-6$

(b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}=-6\mathbf{w}=-6(-4\,,3)=(24\,,-18)$

(c) $\mathbf{u}\cdot(2\mathbf{v})=2(\mathbf{u}\cdot\mathbf{v})=2(-6)=-12$

(d) $\|\mathbf{w}\|^2=\mathbf{w}\cdot\mathbf{w}=(-4)(-4)+(3)(3)=25$

(e) $\mathbf{v}-2\mathbf{w}=(5-(-8)\,,8-6)=(13\,,2)$

$$\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w})=(2)(13)+(-2)(2)=26-4=22$$

- Ex 6: (Using the properties of the dot product)

Given $\mathbf{u}\cdot\mathbf{u}=39$ $\mathbf{u}\cdot\mathbf{v}=-3$ $\mathbf{v}\cdot\mathbf{v}=79$

Find $(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v})$

Sol:

$$
\begin{aligned}
(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(3\mathbf{u}+\mathbf{v})+2\mathbf{v}\cdot(3\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot(3\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+(2\mathbf{v})\cdot(3\mathbf{u})+(2\mathbf{v})\cdot\mathbf{v} \\
&= 3(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+6(\mathbf{v}\cdot\mathbf{u})+2(\mathbf{v}\cdot\mathbf{v}) \\
&= 3(\mathbf{u}\cdot\mathbf{u})+7(\mathbf{u}\cdot\mathbf{v})+2(\mathbf{v}\cdot\mathbf{v}) \\
&= 3(39)+7(-3)+2(79)=254
\end{aligned}
$$

- Thm 5.4: (The Cauchy - Schwarz inequality)

  If **u** and **v** are vectors in R^n, then

  $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ ($|\mathbf{u}\cdot\mathbf{v}|$ denotes the absolute value of $\mathbf{u}\cdot\mathbf{v}$)

- Ex 7: (An example of the Cauchy - Schwarz inequality)

  Verify the Cauchy - Schwarz inequality for **u**=(1, -1, 3) and **v**=(2, 0, -1)

Sol: $\mathbf{u}\cdot\mathbf{v} = -1,\quad \mathbf{u}\cdot\mathbf{u} = 11,\quad \mathbf{v}\cdot\mathbf{v} = 5$

$$\Rightarrow |\mathbf{u}\cdot\mathbf{v}| = |-1| = 1$$

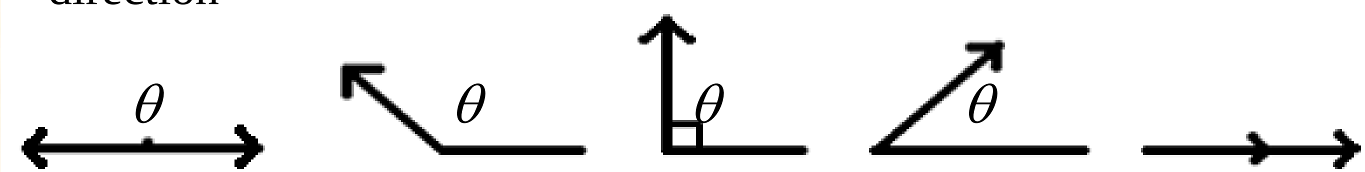
$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}\cdot\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{11}\cdot\sqrt{5} = \sqrt{55}$$

$$\therefore |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

- The angle between two vectors in R^n:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$

Opposite direction	$\mathbf{u}\cdot\mathbf{v} < \mathbf{0}$	$\mathbf{u}\cdot\mathbf{v} = \mathbf{0}$	$\mathbf{u}\cdot\mathbf{v} > \mathbf{0}$	Same direction



$\theta = \pi$	$\frac{\pi}{2} < \theta < \pi$	$\theta = \frac{\pi}{2}$	$0 < \theta < \frac{\pi}{2}$	$\theta = 0$
$\cos = -1$	$\cos < 0$	$\cos = 0$	$\cos > 0$	$\cos = 1$

- Note:

The angle between the zero vector and another vector is not defined.

- Ex 8: (Finding the angle between two vectors)

$$\mathbf{u}=(-4\,,0\,,2\,,-2)\quad \mathbf{v}=(2\,,0\,,-1\,,1)$$

Sol:

$$\|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}}=\sqrt{(-4)^2+0^2+2^2+(-2)^2}=\sqrt{24}$$

$$\|\mathbf{v}\|=\sqrt{\mathbf{v}\cdot\mathbf{v}}=\sqrt{2^2+(0)^2+(-1)^2+1^2}=\sqrt{6}$$

$$\mathbf{u}\cdot\mathbf{v}=(-4)(2)+(0)(0)+(2)(-1)+(-2)(1)=-12$$

$$\Rightarrow \cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{-12}{\sqrt{24}\sqrt{6}}=-\frac{12}{\sqrt{144}}=-1$$

$$\Rightarrow \theta=\pi \qquad \therefore\ \mathbf{u}\text{ and }\mathbf{v}\text{ have opposite directions.}\quad (\mathbf{u}=-2\mathbf{v})$$

- Orthogonal vectors:

  Two vectors **u** and **v** in R^n are orthogonal if

  $$\mathbf{u} \cdot \mathbf{v} = 0$$

- Note:

  The vector **0** is said to be orthogonal to every vector.

- Ex 10: (Finding orthogonal vectors)

Determine all vectors in R^n that are orthogonal to **u**=(4, 2).

Sol:

$$\mathbf{u} = (4\,,2) \quad \text{Let} \quad \mathbf{v} = (v_1\,,v_2)$$

$$\Rightarrow \quad \mathbf{u}\cdot\mathbf{v} = (4\,,2)\cdot(v_1\,,v_2)$$

$$= 4v_1 + 2v_2$$

$$= 0$$

$$\begin{bmatrix}4 & 2 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}$$

$$\Rightarrow \quad v_1 = \frac{-t}{2} \quad , \quad v_2 = t$$

$$\therefore \quad \mathbf{v} = \left(\frac{-t}{2}, t\right), \qquad t \in R$$

- Thm 5.5: (The triangle inequality)

  If **u** and **v** are vectors in R^n, then $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

Pf:

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$$

$$= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \quad \le \|\mathbf{u}\|^2 + 2\,|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2$$

$$\le \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2$$

$$= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

$$\therefore \|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

- Note:

  Equality occurs in the triangle inequality if and only if the vectors **u** and **v** have the same direction.

- Thm 5.6: (The Pythagorean theorem)

If **u** and **v** are vectors in R^n, then **u** and **v** are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$$

- Dot product and matrix multiplication:

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}\qquad \mathbf{v}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$$

(A vector $\mathbf{u}=(u_1\,,u_2\,,\cdots,u_n)$ in R^n is represented as an $n\times 1$ column matrix)

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=[u_1\quad u_2\quad\cdots\quad u_n]\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=[u_1v_1+u_2v_2+\cdots+u_nv_n]$$

5.4 Cross Product

- Cross product in R^3:

  The **cross product** of $\mathbf{u}=(u_1, u_2, u_3)$ and $\mathbf{v}=(v_1, v_2, v_3)$

  is the vector quantity

$$\mathbf{w}=\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}=\left(\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}, -\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}, \begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\right)$$

$$\mathbf{w}=\mathbf{u}\times\mathbf{v}=\left(u_2v_3-u_3v_2, u_3v_1-u_1v_3, u_1v_2-u_2v_1\right)$$

- Ex 11: (Finding the cross product of two vectors)

  The cross product of **u**=(1, 2, 0) and **v**=(3, -2, 4) is

$$\mathbf{w}=\mathbf{u}\times\mathbf{v}=\left(8,-4,-8\right)$$

- Thm 5.10: Relationships involving cross product and dot product

Let **u, v** and **w** be 3 vectors in R^3, then:

(a)	$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=0$	($\mathbf{u}\times\mathbf{v}$ is *orthogonal to* $\mathbf{u}$)
(b)	$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=0$	($\mathbf{u}\times\mathbf{v}$ *is orthogonal to* $\mathbf{v}$)
(c)	$\lVert\mathbf{u}\times\mathbf{v}\rVert^2=\lVert\mathbf{u}\rVert^2\lVert\mathbf{v}\rVert^2-(\mathbf{u}\cdot\mathbf{v})^2$	(*Lagrange ' s identity*)
(d)	$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$	(*relationship between cross and dot products*)
(e)	$(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}$	(*relationship between cross and dot products*)

- Thm 5.11: Properties of involving cross product

Let **u, v** and **w** be 3 vectors in R^3 and *k* a scalar, then:

(a) $\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$

(b) $\mathbf{u}\times(\mathbf{v}+\mathbf{w})=(\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$

(c) $(\mathbf{u}+\mathbf{v})\times\mathbf{w}=(\mathbf{u}\times\mathbf{w})+(\mathbf{v}\times\mathbf{w})$

(d) $k(\mathbf{u}\times\mathbf{v})=(k\mathbf{u})\times\mathbf{v}=\mathbf{u}\times(k\mathbf{v})$

(e) $\mathbf{u}\times\mathbf{0}=\mathbf{0}\times\mathbf{u}=\mathbf{0}$

(f) $\mathbf{u}\times\mathbf{u}=\mathbf{0}$

- Thm 5.12: Scalar triple product

Let **u, v** and **w** be 3 vectors in R^3, then:

$$V = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \text{Volume of the parallelepiped determined by the 3 vectors}$$