Chapter 4

Vector Spaces الفضاءات الاتجاهية

4.1 Vectors in R^n

- An ordered *n*-tuple:

  a sequence of *n* real number $(x_1, x_2, \cdots, x_n)$

- *n*-space: R^n

  the set of all ordered n-tuple

- Ex:

$n = 1$ R^1 = 1-space
= set of all real number

$n = 2$ R^2 = 2-space
= set of all ordered pair of real numbers (x_1, x_2)

$n = 3$ R^3 = 3-space

= set of all ordered triple of real numbers (x_1, x_2, x_3)

$n = 4$ R^4 = 4-space

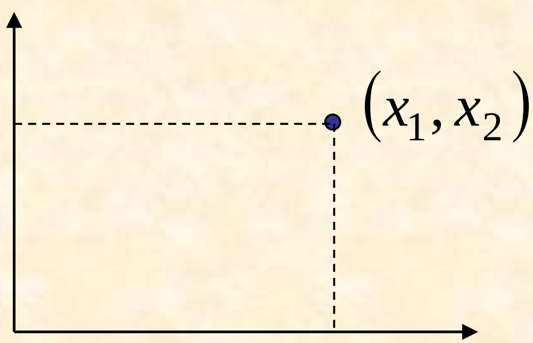
= set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

- Notes:

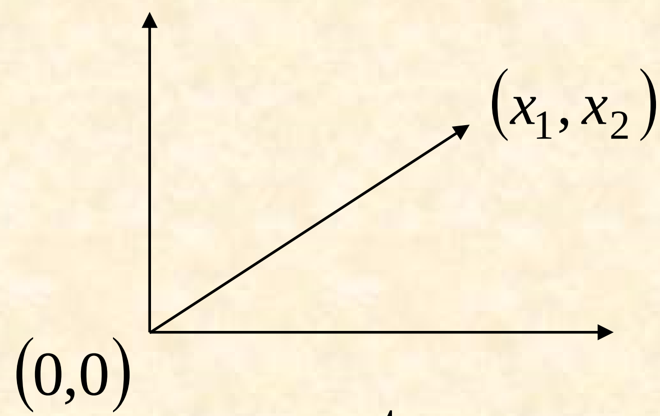
(1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a point</u> in R^n with the x_i's as its coordinates.

(2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a vector</u> $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.

- Ex:



a point



a vector

4.2 Vector Spaces

- Vector spaces:

Let V be a set on which two operations (*vector addition and scalar multiplication*) are defined. If the following axioms are satisfied for every **u**, **v**, and **w** in V and every scalar (real number) c and d, then V is called a **vector space**.

Addition:

(1) $\mathbf{u}+\mathbf{v}$ is in V *too* (***closed*** *under addition)*

(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

(4) V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u}+\mathbf{0}=\mathbf{u}$

(5) For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in V *too* (**closed** *under multiplication by a scalar).*

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- **Note:**

(1) A vector space consists of **four entities**:

a set of vectors, a set of scalars, and two operations

V : nonempty set

c : scalar

$+ \quad (\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v}$ vector addition

$\bullet \quad (c, \mathbf{u}) = c\mathbf{u}$ scalar multiplication

$(V, \ +, \ \bullet)$ **is then called a vector space**

(2) $V = \{\mathbf{0}\}$: zero vector space

- Examples of vector spaces:

(1) n-tuple space: $\boxed{R^n}$

$(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$ vector addition

$k(u_1, u_2, \cdots, u_n) = (ku_1, ku_2, \cdots, ku_n)$ scalar multiplication

(2) Matrix space: $\boxed{V = M_{m\times n}}$ (the set of all $m\times n$ matrices with real values)

Exp: ：$(m = n = 2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

(3) n-th degree polynomial space: $\boxed{V = P_n(x)}$

(the set of all real polynomials of degree n or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

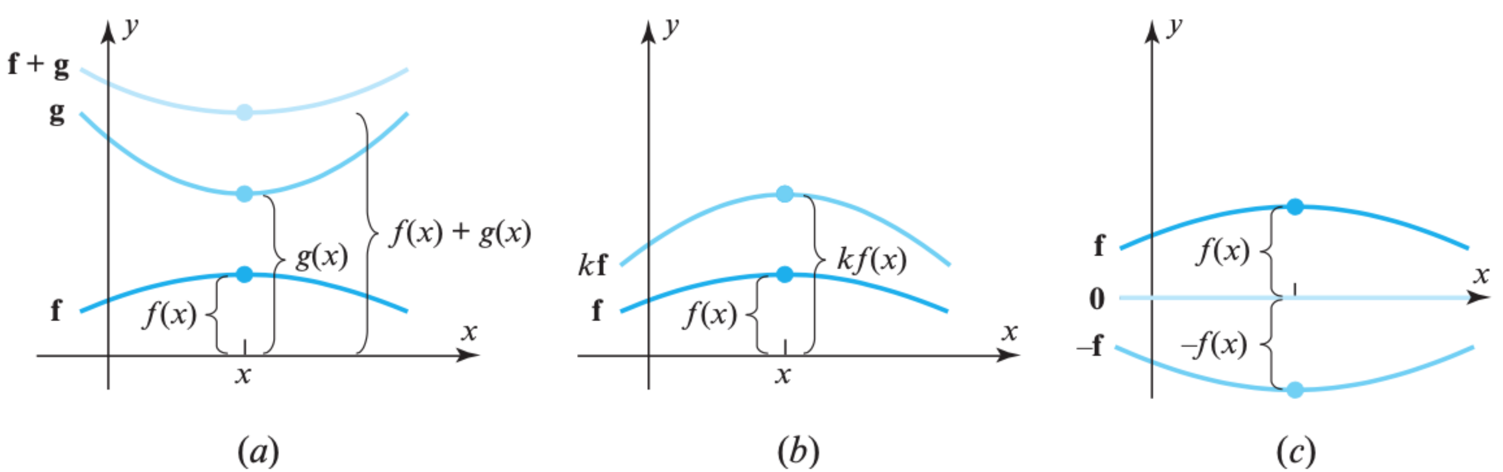
$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$

(4) Function space: $\boxed{V = c(-\infty,\infty)}$ (the set of all real-valued continuous functions defined on the entire real line.)

$$(f+g)(x)=f(x)+g(x)$$

$$(kf)(x)=kf(x)$$

$$(f+g)(x) = f(x) + g(x) \qquad (kf)(x) = kf(x)$$



▲ Figure 4.1.2

- Thm 4.4: (Properties of scalar multiplication)

  Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

(1) $0\mathbf{v} = \mathbf{0}$

(2) $c\mathbf{0} = \mathbf{0}$

(3) If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$

(4) $(-1)\mathbf{v} = -\mathbf{v}$

- Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

- Ex 6: The set of all integer is not a vector space.

Pf:

$$1 \in V, \tfrac{1}{2} \in R$$

$$\underset{\text{scalar}}{\left(\tfrac{1}{2}\right)}\underset{\text{integer}}{(1)} = \underset{\text{noninteger}}{\tfrac{1}{2}} \notin V$$

(it is not closed under scalar multiplication)

- Ex 7: The set of all second-degree polynomials is not a vector space.

Pf: Let $p(x) = x^2$ and $q(x) = -x^2 + x + 1$

$$\Rightarrow p(x) + q(x) = x + 1 \notin V$$

(it is not closed under vector addition)

- Ex 8:

$V=R^2$=the set of all ordered pairs of real numbers **defined as:**

- vector addition: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$

- scalar multiplication: $c(u_1,u_2)=(cu_1,0)$

---------Verify V is not a vector space.

Sol:

$\because 1(1,1)=(1,0)\neq(1,1)$ Condition (10) is not satisfied

$\therefore$ the set (together with the two given operations) **is not** a vector space

▶ EXAMPLE 7 A Set That Is Not a Vector Space

Let $V = R^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0)$$

For example, if $\mathbf{u} = (2, 4)$, $\mathbf{v} = (-3, 5)$, and $k = 7$, then

$$\mathbf{u} + \mathbf{v} = (2 + (-3), 4 + 5) = (-1, 9)$$

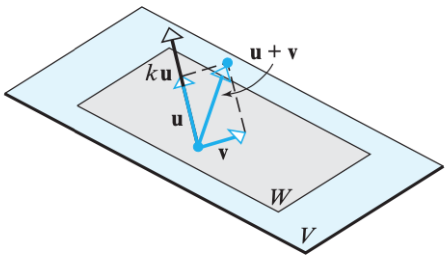
$$k\mathbf{u} = 7\mathbf{u} = (7 \cdot 2, 0) = (14, 0)$$

The addition operation is the standard one from R^2, but the scalar multiplication is not. In the exercises we will ask you to show that the first nine vector space axioms are satisfied. However, Axiom 10 fails to hold for certain vectors. For example, if $\mathbf{u} = (u_1, u_2)$ is such that $u_2 \neq 0$, then

$$1\mathbf{u} = 1(u_1, u_2) = (1 \cdot u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

Thus, V is not a vector space with the stated operations. ◀

- Subspace:



▶ Figure 4.2.1 The vectors **u** and **v** are in W, but the vectors **u** + **v** and k**u** are not.

If

$(V,+,\bullet)$: a vector space

$\left.\begin{array}{l} W \neq \phi \\ W \subseteq V \end{array}\right\}$: a nonempty subset

$(W,+,\bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ *Then* ***W*** **is a subspace** of ***V***

- Trivial subspace:

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

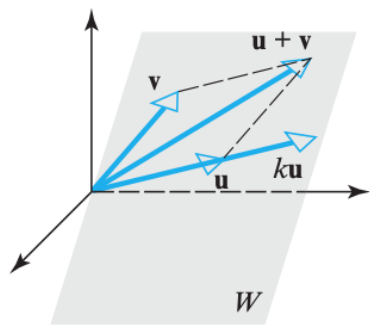
(2) V is a subspace of V.

- Thm 4.5: (Test for a subspace)

  If W is a **nonempty subset** of a vector space V, then W is a subspace of V **if and only if** the following conditions hold.

  (1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W (*closed under addition*).

  (2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W (*closed under multiplication by a scalar*).



▲ Figure 4.2.3 The vectors $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ both lie in the same plane as $\mathbf{u}$ and $\mathbf{v}$.

- Ex: Subspace of R^2

  (1) $\{\mathbf{0}\}$ $\quad \mathbf{0} = (0, 0)$

  (2) Lines through the origin

  (3) R^2

- Ex: Subspace of R^3

  (1) $\{\mathbf{0}\}$ $\quad \mathbf{0} = (0, 0, 0)$

  (2) Lines through the origin

  (3) Planes through the origin

  (4) R^3

Table 1

Subspaces of R^2	Subspaces of R^3
• $\{0\}$ • Lines through the origin • R^2	• $\{0\}$ • Lines through the origin • Planes through the origin • R^3

- Ex 2: (A subspace of $M_{2\times 2}$)

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times 2}$, with the standard operations of matrix addition and scalar multiplication.

Sol:

$\because W \subseteq M_{2\times 2} \quad M_{2\times 2}:\text{vector sapces}$

$\text{Let } A_1, A_2 \in W \ (A_1^T = A_1, A_2^T = A_2)$

$A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2$ $(A_1 + A_2 \in W)$

$k \in R, A \in W \Rightarrow (kA)^T = kA^T = kA$ $(kA \in W)$

$\therefore W \text{ is a subspace of } M_{2\times 2}$

- Ex 3: (The set of singular matrices is not a subspace of $M_{2\times 2}$)

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times 2}$ with the standard operations.

Sol:

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in W, B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\in W$$

$$\therefore A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\notin W$$

$$\therefore W_2 \text{ is not a subspace of } M_{2\times 2}$$

singular matrices

The matrices are known to be singular if their determinant is equal to the zero

- Ex 4: (The set of first-quadrant vectors is not a subspace of R^2)

Show that $W=\{(x_1,x_2): x_1\geq 0 \text{ and } x_2\geq 0\}$, with the standard operations, is not a subspace of R^2.

(not closed under scalar multiplication)

Sol:

Let $\mathbf{u}=(1,1)\in W$

$\because (-1)\mathbf{u}=(-1)(1,1)=(-1,-1)\notin W$

$\therefore W$ is not a subspace of R^2

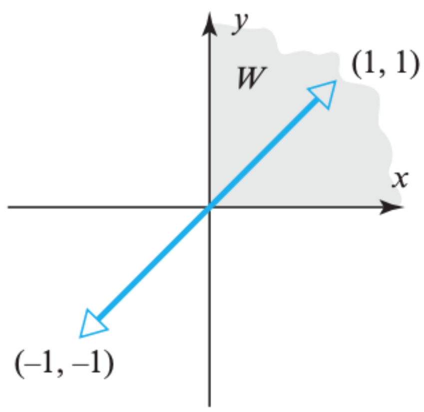


Figure 4.2.4 W is not closed under scalar multiplication.

4.4 Spanning Sets and Linear Independence

- Linear combination:

A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$

- Ex 2-3: (Finding a linear combination)

$$\mathbf{v}_1=(1,2,3)\quad \mathbf{v}_2=(0,1,2)\quad \mathbf{v}_3=(-1,0,1)$$

Prove (a) $\mathbf{w}=(1,1,1)$ is a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

(b) $\mathbf{w}=(1,-2,2)$ is not a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

Sol:

(a) $\mathbf{w}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$

$$\begin{aligned}(1,1,1)&=c_1(1,2,3)+c_2(0,1,2)+c_3(-1,0,1)\\&=(c_1-c_3,\ 2c_1+c_2,\ 3c_1+2c_2+c_3)\end{aligned}$$

$$\Rightarrow\begin{array}{rcrcrl} c_1 & & & - & c_3 & =1\\ 2c_1 & + & c_2 & & & =1\\ 3c_1 & + & 2c_2 & + & c_3 & =1\end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass}-\text{Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t\,,\ c_2 = -1 - 2t\,,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass−Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ this system has no solution $(\because 0 \neq 7)$

$$\Rightarrow \mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

- the span of a set: span (S)

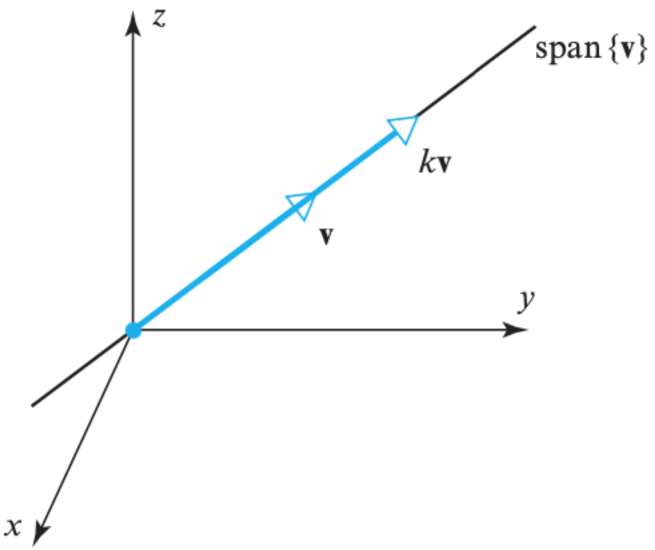
If $S=\{\mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

$$\boxed{span(S)=\{c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k \mid \forall c_i\in R\}}$$

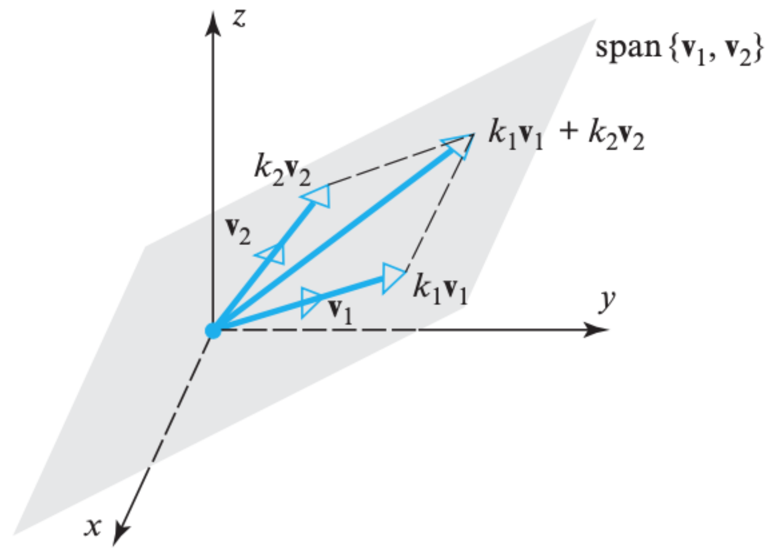
(the set of all linear combinations of vectors in S)

- a spanning set of a vector space:

If **every vector** in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.



(*a*) Span $\{\mathbf{v}\}$ is the line through the origin determined by $\mathbf{v}$.



(*b*) Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

- Notes:

$$\text{span}\,(S) = V$$

$$\Rightarrow\ S \text{ spans (generates) } V$$

$$V \text{ is spanned (generated) by } S$$

$$S \text{ is a spanning set of } V$$

- Notes:

(1) $span(\phi) = \{\mathbf{0}\}$

(2) $S \subseteq span(S)$

(3) $S_1, S_2 \subseteq V$

$$S_1 \subseteq S_2 \Rightarrow span(S_1) \subseteq span(S_2)$$

- Linear Independence **(L.I.)** and Linear Dependence (**L.D.**):

$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$: a set of vectors in a vector space V

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

(1) If the equation has only the trivial solution $(c_1 = c_2 = \cdots = c_k = 0)$
then S is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all zeros),
then S is called linearly dependent.

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- Notes:

(1) ϕ is linearly independent

(2) $\mathbf{0} \in S \Rightarrow S$ is linearly dependent.

(3) $\mathbf{v} \neq \mathbf{0} \Rightarrow \{\mathbf{v}\}$ is linearly independent

(4) $S_1 \subseteq S_2$

S_1 is linearly dependent $\Rightarrow S_2$ is linearly dependent

S_2 is linearly independent $\Rightarrow S_1$ is linearly independent

- Ex 8: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

$$\qquad\quad \mathbf{v}_1 \qquad\quad \mathbf{v}_2 \qquad\qquad \mathbf{v}_3$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \;\Rightarrow\; \begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + \; c_2 + \qquad\; &= 0 \\ 3c_1 + 2c_2 + \; c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

$$\Rightarrow S \text{ is linearly independent}$$

- Ex 10: (Testing for linearly independent)

Determine whether the following set of vectors in 2×2 matrix space is L.I. or L.D.

$$S = \left\{ \underset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2c_1+3c_2+\ \ c_3 &= 0 \\ c_1 \qquad\qquad\quad &= 0 \\ 2c_2+2c_3 &= 0 \\ c_1+c_2 \qquad\quad &= 0 \end{aligned}$$

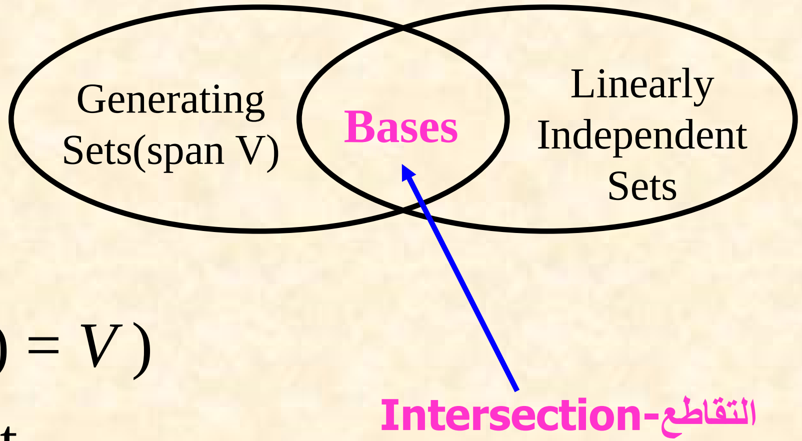
$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution.)

$\Rightarrow$ S is linearly independent.

4.5 Basis and Dimension

- Basis:



V : a vector space

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$

$$\underline{\textbf{If}} \begin{cases} (a) & S \text{ spans } V \text{ (i.e., } span(S) = V \text{)} \\ (b) & S \text{ is linearly independent} \end{cases}$$

$\Rightarrow$ *Then* S is called a **basis** for V

- Notes:

(1) Ø is a basis for $\{\mathbf{0}\}$

(2) the standard basis for R^3:

$\{i, j, k\} \quad i = (1, 0, 0),\ j = (0, 1, 0),\ k = (0, 0, 1)$

(3) the standard basis for $\boldsymbol{R}^n$:

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\quad \mathbf{e}_1=(1,0,\ldots,0)$, $\mathbf{e}_2=(0,1,\ldots,0)$, $\mathbf{e}_n=(0,0,\ldots,1)$

Ex: R^4 $\quad \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

(4) the standard basis for $m\times n$ matrix space:

$\{ E_{ij} \mid 1\le i\le m , 1\le j\le n \}$

Ex: for $\mathbf{M}_{22}$ 2×2 matrix space:

$$\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$$

(5) the standard basis for $\boldsymbol{P_n(x)}$:

$\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x)$ $\quad \{1, x, x^2, x^3\}$

- Thm 4.9: (Uniqueness of basis representation)

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Pf:

$$\because S \text{ is a basis} \Rightarrow \begin{cases} 1. & span(S) = V \\ 2. & S \text{ is linearly independent} \end{cases}$$

$\because span(S) = V$ Let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$

$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n$

$\Rightarrow \mathbf{0} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \ldots + (c_n - b_n)\mathbf{v}_n$

$\because S$ is linearly independent

$\Rightarrow c_1 = b_1,\ c_2 = b_2, \ldots, c_n = b_n$ (i.e., uniqueness)

- Thm 4.10: (Basis and linear dependence)

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

Pf:

Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, $m > n$

$\because span(S)=V$

$$\mathbf{u}_i \in V \quad \Rightarrow \quad \begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdots + c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \cdots + c_{nm}\mathbf{v}_n \end{aligned}$$

Let $k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$

$\Rightarrow d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_n\mathbf{v}_n=\mathbf{0}$ (where $d_i=c_{i1}k_1+c_{i2}k_2+\ldots+c_{im}k_m$)

$\because S$ is L.I.

$\Rightarrow d_i=0 \quad \forall i$ i.e.

$$\begin{aligned} c_{11}k_1+c_{12}k_2+\cdots+c_{1m}k_m &= 0 \\ c_{21}k_1+c_{22}k_2+\cdots+c_{2m}k_m &= 0 \\ &\vdots \\ c_{n1}k_1+c_{n2}k_2+\cdots+c_{nm}k_m &= 0 \end{aligned}$$

$\because$ According to Thm 1.1: If the homogeneous system has fewer equations than variables, then it must have infinitely many solution.

$m>n \Rightarrow k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$ has nontrivial solution

$\Rightarrow S_1$ is linearly dependent

- Thm 4.11: (Number of vectors in a basis)

  If a vector space V has one basis with ***n* vectors**, then every basis for V has n vectors. (*All bases for a finite-dimensional vector space has the same number of vectors*.)

Pf:

$S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

$S'=\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$

two bases for a vector space

$$\left.\begin{array}{l}\left.\begin{array}{r}S \text{ is a basis}\\ S' \text{ is L.I.}\end{array}\right\} \overset{Thm.4.10}{\Rightarrow} n \geq m \\ \left.\begin{array}{r}S \text{ is L.I.}\\ S' \text{ is a basis}\end{array}\right\} \overset{Thm.4.10}{\Rightarrow} n \leq m\end{array}\right\} \Rightarrow n = m$$

- Finite dimensional:

  A vector space V is called **finite dimensional**,

  if it has a basis consisting of a finite number of elements.

- Infinite dimensional:

  If a vector space V is not finite dimensional,

  then it is called **infinite dimensional**.

- Dimension:

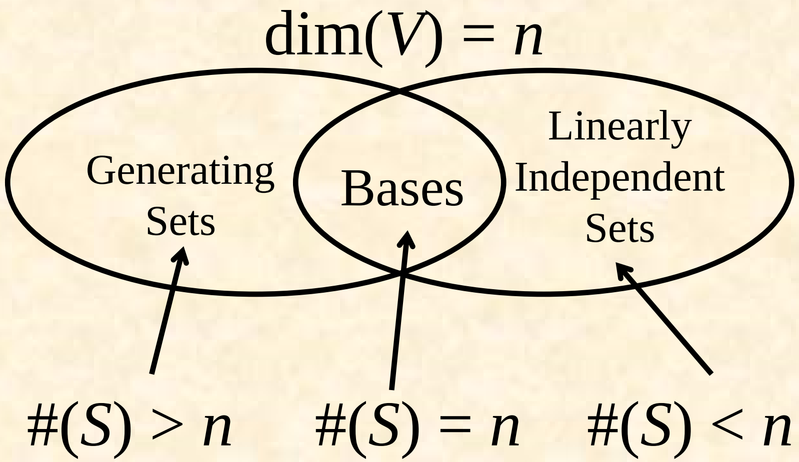
  The **dimension** of a finite dimensional vector space V is

  defined to be **the number of vectors in a basis** for V.

  V: a vector space S: a basis for V

  $\Rightarrow$symbol: $\dim(V) = \#(S)$ (the number of vectors in S)

- Notes:



(1) $\dim(\{\mathbf{0}\}) = 0 = \#(\varnothing)$

(2) $\dim(V) = n$, $S \subseteq V$

S：a generating set $\Rightarrow \#(S) \geq n$

S：a L.I. set $\Rightarrow \#(S) \leq n$

S：a basis $\Rightarrow \#(S) = n$

(3) $\dim(V) = n$, W is a subspace of V $\Rightarrow \dim(W) \leq n$

- Exp:

(1) Vector space R^n $\Rightarrow$ basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$\Rightarrow \dim(R^n) = n$

(2) Vector space $M_{m\times n}$ $\Rightarrow$ basis $\{E_{ij} \mid 1\le i\le m,\ 1\le j\le n\}$

$\Rightarrow \dim(M_{m\times n})=mn$

(3) Vector space $P_n(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots, x^n\}$

$\Rightarrow \dim(P_n(x)) = n+1$

(4) Vector space $P(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots\}$

$\Rightarrow \dim(P(x)) = \infty$

- Ex 9: (Finding the dimension of a subspace)

(a) $W=\{(d, c-d, c)$: c and d are real numbers$\}$

(b) $W=\{(2b, b, 0)$: b is a real number$\}$

Sol: (Note: Find a set of L.I. vectors that spans the subspace)

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S *spans* W)

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2$

(b) $\because (2b, b, 0) = b(2,1,0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$

- Ex 10: (Finding the dimension of a subspace)

  Let W be the subspace of all symmetric matrices in $M_{2\times 2}$.

  What is the dimension of W?

Sol:

$$W=\left\{\begin{bmatrix} a & b \\ b & c \end{bmatrix}\middle| a,b,c\in R\right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix}=a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow S=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W $\Rightarrow$ dim(W) = #(S) = 3

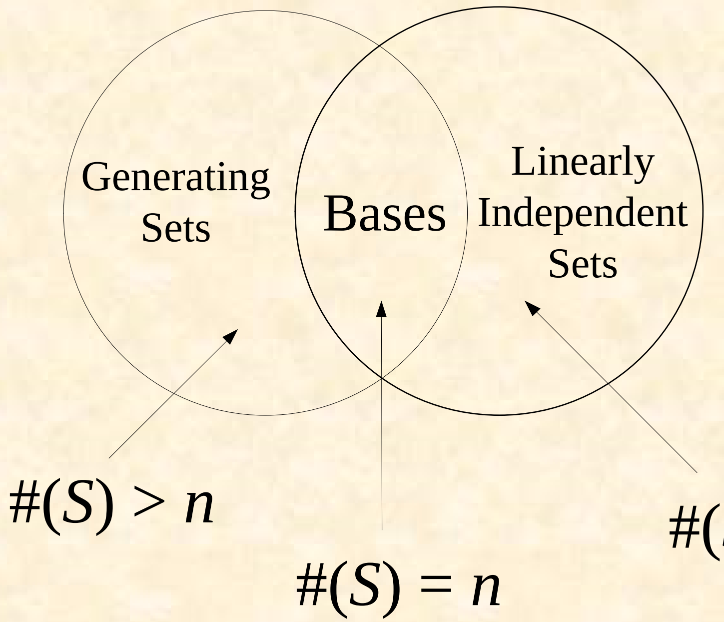
- Thm 4.12: (Basis tests in an n-dimensional space)

  Let V be a vector space of **dimension *n***.

  (1) If $S=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.

  (2) If $S=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ spans V, then S is a basis for V.

$$\dim(V) = n$$



Imp: If we have a space V of dimension n, and a set of vectors S of number equal n, then for the set of vectors S to be a Basis of V, it is sufficient to show that S is L.I. or that it spans V.

Our next goal is to extend the concepts of "basis vectors" and "coordinate systems" to general vector spaces, and for that purpose we will need some definitions. Vector spaces fall into two categories: A vector space V is said to be ***finite-dimensional*** if there is a finite set of vectors in V that spans V and is said to be ***infinite-dimensional*** if no such set exists.

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a ***basis*** for V if:

(a) S spans V.

(b) S is linearly independent.

EXAMPLE 9 Coordinates in R^3

(a) We showed in Example 3 that the vectors

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (2, 9, 0), \quad \mathbf{v}_3 = (3, 3, 4)$$

form a basis for R^3. Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Find the vector $\mathbf{v}$ in R^3 whose coordinate vector relative to S is $(\mathbf{v})_S = (-1, 3, 2)$.

Solution (a) To find $(\mathbf{v})_S$ we must first express $\mathbf{v}$ as a linear combination of the vectors in S; that is, we must find values of c_1, c_2, and c_3 such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

or, in terms of components,

$$(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$$

Equating corresponding components gives

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 + 4c_3 &= 9 \end{aligned}$$

Solving this system we obtain $c_1 = 1$, $c_2 = -1$, $c_3 = 2$ (verify). Therefore,

$$(\mathbf{v})_S = (1, -1, 2)$$

Solution (b) Using the definition of $(\mathbf{v})_S$, we obtain

$$\begin{aligned} \mathbf{v} &= (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 \\ &= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) = (11, 31, 7) \end{aligned}$$ ◀