Chapter 3

Determinants المحددات

3.1 The Determinant of a Matrix

3.2 Evaluation of a Determinant using Elementary Operations

3.3 Properties of Determinants

3.4 Application of Determinants

3.1 The Determinant of a Matrix

- the determinant of a 2 × 2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- Note: Symbol

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- Ex. 1: (The determinant of a matrix of order 2)

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) \quad = 4 + 3 \quad = 7$$

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) \quad = 4 - 4 \quad = 0$$

$$\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) \quad = 0 - 6 \quad = -6$$

- Note: The determinant of a matrix can be positive, zero, or negative.

- Minor of the entry a_{ij} :

  The determinant of the matrix determined by <u>deleting the *i*th row and *j*th column of A</u>

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & & \vdots & \vdots & & \\ a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_{n1} & & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

- We define the ***<u>Cofactor</u>*** of the entry a_{ij} :

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Called Cofactor

Called Minor
(determinant by deleting the ***i***th *row and* ***j***th *column)*

- Ex:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21}=\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{22}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21}=(-1)^{2+1}M_{21}=-M_{21} \qquad C_{22}=(-1)^{2+2}M_{22}=M_{22}$$

- Notes: Sign pattern for cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

3×3 matrix $\qquad$ 4×4 matrix $\qquad$ $n \times n$ matrix

- Notes:

  Odd positions (where $i+j$ is odd) have negative signs, and even positions (where $i+j$ is even) have positive signs.

- Ex 2: Find all the minors and cofactors of *A*.

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Sol: (1) All the minors of *A (9 minors)*.

$$\Rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \; M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5, \; M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2, \quad M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \; M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \quad M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3, \; M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

Sol: (2) All the cofactors of A *(9 cofactors)*.

$$\because \quad C_{ij} = (-1)^{i+j} M_{ij}$$

$$\Rightarrow C_{11} = +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \; C_{12} = -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5, \quad C_{13} = +\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2, \quad C_{22} = +\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \; C_{23} = -\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = +\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \quad C_{32} = -\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3, \quad C_{33} = +\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

- Thm 3.1: (Expansion by cofactors)

Let A is a **square matrix** of order n.

Then the determinant of A is given by

$$(a) \quad \det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(Cofactor expansion along the i-th row, i=1, 2,…, n)

<u>or</u>

$$(b) \quad \det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(Cofactor expansion along the j-th column, j=1, 2,…, n)

<u>Rk:</u> expansion along any row or column

- Ex: The determinant of a matrix of order 3

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}\Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}\end{aligned}$$

- Ex 3: The determinant of a matrix of order 3

$$A=\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \overset{\text{Ex 2}}{\Rightarrow} \begin{matrix} C_{11}=-1, C_{12}=5, \quad C_{13}=4 \\ C_{21}=-2, C_{22}=-4, C_{23}=8 \\ C_{31}=5, \quad C_{32}=3, \quad C_{33}=-6 \end{matrix}$$

Sol:

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13} = (0)(-1)+(2)(5)+(1)(4)=14 \\ &= a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23} = (3)(-2)+(-1)(-4)+(2)(8)=14 \\ &= a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33} = (4)(5)+(0)(3)+(1)(-6)=14 \\ &= a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31} = (0)(-1)+(3)(-2)+(4)(5)=14 \\ &= a_{12}C_{12}+a_{22}C_{22}+a_{32}C_{32} = (2)(5)+(-1)(-4)+(0)(3)=14 \\ &= a_{13}C_{13}+a_{23}C_{23}+a_{33}C_{33} = (1)(4)+(2)(8)+(1)(-6)=14 \end{aligned}$$

- Ex 5: (The determinant of a matrix of order 3)

$$A=\begin{bmatrix}0 & 2 & 1\\ 3 & -1 & 2\\ 4 & 0 & 1\end{bmatrix} \Rightarrow \det(A)=?$$

Sol:

$$C_{11}=(-1)^{1+1}\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=-1 \qquad C_{12}=(-1)^{1+2}\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}=(-1)(-5)=5$$

$$C_{13}=(-1)^{1+3}\begin{vmatrix}3 & -1\\ 4 & 0\end{vmatrix}=4$$

$$\begin{aligned}\Rightarrow \det(A) &= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}\\ &=(0)(-1)+(2)(5)+(1)(4)\\ &=14\end{aligned}$$

- Notes:

  The row (or column) containing the most zeros **is the best choice** for expansion by cofactors .

- Ex 4: (The determinant of a matrix of order 4)

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43})$$

$$= 3C_{13}$$

$$= 3(-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3\left[(0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}\right]$$

$$= 3[0 + (2)(1)(-4) + (3)(-1)(-7)]$$

$$= (3)(13)$$

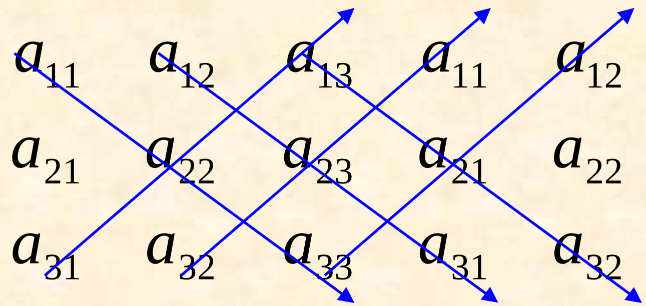
$$= 39$$

- The determinant of a matrix of **order 3**:

Called the *Arrow* Method

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Subtract these three products.



Add these three products.

$$\Rightarrow \det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13}$$
$$- a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

- Ex 5: (طريقة لحساب سريع تهم فقط المصفوفة الثلاثية)

Called the *Arrow* Method

$$\begin{matrix} & & & -4 & 0 & 6 \end{matrix}$$

$$A=\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}\begin{matrix} 0 & 2 \\ 3 & -1 \\ 4 & -4 \end{matrix}$$

$$\begin{matrix} & & & 0 & 16 & -12 \end{matrix}$$

$$\Rightarrow \det(A)=|A|=0+16-12-(-4)-0-6=2$$

- Upper triangular matrix:

  All the entries below the main diagonal are zeros.

- Lower triangular matrix:

  All the entries above the main diagonal are zeros.

- Diagonal matrix:

  All the entries above and below the main diagonal are zeros.

- Note:

  A matrix that is both upper and lower triangular is called diagonal.

- Ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular   lower triangular   diagonal

- Thm 3.2: (Determinant of a Triangular Matrix)

  If A is an $n \times n$ triangular matrix (**upper triangular, lower triangular, or diagonal**), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$$

- Ex 6: Find the determinants of the following triangular matrices.

$$(a)\quad A=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix} \qquad (b)\quad B=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$(a)\quad |A| = (2)(-2)(1)(3) = -12$

$(b)\quad |B| = (-1)(3)(2)(4)(-2) = 48$

Keywords in Section 3.1:

- determinant : المحدد
- minor : المختصر
- cofactor : المعامل
- expansion by cofactors : التحليل بالمعاملات
- upper triangular matrix: مصفوفة مثلثية سفلى
- lower triangular matrix: مصفوفة مثلثية عليا
- diagonal matrix: مصفوفة قطرية

- Thm 3.3: (Elementary row operations and determinants)

Let A and B be square matrices.

(a) $B = r_{ij}(A) \Rightarrow \det(B) = -\det(A)$ (i.e. $|r_{ij}(A)| = -|A|$)

(b) $B = r_i^{(k)}(A) \Rightarrow \det(B) = k\det(A)$ (i.e. $|r_i^{(k)}(A)| = k|A|$)

(c) $B = r_{ij}^{(k)}(A) \Rightarrow \det(B) = \det(A)$ (i.e. $|r_{ij}^{(k)}(A)| = |A|$)

العمليات

تأثير عمليات السطر البسيطة على المحددة

- Ex: knowing that A and its determinant are given

$$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 1 & 2 & 1\end{bmatrix} \qquad \det(A)=-2$$

Find the Determinant of

$$A_1=\begin{bmatrix}4 & 8 & 12\\ 0 & 1 & 4\\ 1 & 2 & 1\end{bmatrix} \quad A_2=\begin{bmatrix}0 & 1 & 4\\ 1 & 2 & 3\\ 1 & 2 & 1\end{bmatrix} \quad A_3=\begin{bmatrix}1 & 2 & 3\\ -2 & -3 & -2\\ 1 & 2 & 1\end{bmatrix}$$

$$A_1=r_1^{(4)}(A) \quad \Rightarrow \det(A_1)=\det(r_1^{(4)}(A))=4\det(A)=(4)(-2)=-8$$

$$A_2=r_{12}(A) \quad \Rightarrow \det(A_2)=\det(r_{12}(A))=-\det(A)=-(-2)=2$$

$$A_3=r_{12}^{(-2)}(A) \quad \Rightarrow \det(A_3)=\det(r_{12}^{(-2)}(A))=\det(A)=-2$$

- Notes:

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k\det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k}\det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$

Note:

A row-echelon form of a square matrix is always *upper triangular*.

- Ex 2: (Evaluation a determinant using elementary row operations)

$$A=\begin{bmatrix}2 & -3 & 10\\ 1 & 2 & -2\\ 0 & 1 & -3\end{bmatrix} \Rightarrow \det(A)=?$$

تبديل سطرين

Sol:

$$\det(A)=\begin{vmatrix}2 & -3 & 10\\ 1 & 2 & -2\\ 0 & 1 & -3\end{vmatrix} \overset{r_{12}}{=} -\begin{vmatrix}1 & 2 & -2\\ 2 & -3 & 10\\ 0 & 1 & -3\end{vmatrix}$$

$$\overset{r_{12}^{(-2)}}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_2^{(-\frac{1}{7})}}{=} (-1)\left(\frac{1}{\frac{-1}{7}}\right)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix}$$

لاتغير في المحددة

$$\overset{r_{23}^{(-1)}}{=} 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = (7)(1)(1)(-1) = -7$$

يجب الضرب في مقلوب العدد

- Notes:

$$|EA| = |E||A|$$ (E: elementary matrix أي ممكن ايجاد *I* من عملية سطر بسيطة واحدة)

$$If\,(1) \quad E = R_{ij} \quad \Rightarrow |E| = |R_{ij}| = -1$$

$$\Rightarrow |EA| = |r_{ij}(A)| = -|A| = |R_{ij}||A| = |E||A|$$

$$If\,(2) \quad E = R_i^{(k)} \Rightarrow |E| = |R_i^{(k)}| = k$$

$$\Rightarrow |EA| = |r_i^{(k)}(A)| = k|A| = |R_i^{(k)}||A| = |E||A|$$

$$If\,(3) \quad E = R_{ij}^{(k)} \Rightarrow |E| = |R_{ij}^{(k)}| = 1$$

$$\Rightarrow |EA| = |r_{ij}^{(k)}(A)| = 1|A| = |R_{ij}^{(k)}||A| = |E||A|$$

- Determinants and elementary column operations

- Thm: (Elementary **column operations** and determinants)

  Let A and B be square matrices.

$$(a)\quad B = c_{ij}(A) \;\Rightarrow\; \det(B) = -\det(A) \qquad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

$$(b)\quad B = c_{i}^{(k)}(A) \;\Rightarrow\; \det(B) = k\det(A) \qquad (\text{i.e. } |c_{i}^{(k)}(A)| = k|A|)$$

$$(c)\quad B = c_{ij}^{(k)}(A) \;\Rightarrow\; \det(B) = \det(A) \qquad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$

تبقى صحيحة
مثل عمليات الأسطر

- Ex:

$$A=\begin{bmatrix}2 & 1 & -3\\ 4 & 0 & 1\\ 0 & 0 & 2\end{bmatrix} \qquad \det(A)=-8$$

- Find ***det*** of:

$$A_1=\begin{bmatrix}1 & 1 & -3\\ 2 & 0 & 1\\ 0 & 0 & 2\end{bmatrix} \quad A_2=\begin{bmatrix}1 & 2 & -3\\ 0 & 4 & 1\\ 0 & 0 & 2\end{bmatrix} \quad A_3=\begin{bmatrix}2 & 1 & 0\\ 4 & 0 & 1\\ 0 & 0 & 2\end{bmatrix}$$

$$A_1=c_1^{(\frac{1}{2})}(A) \Rightarrow \det(A_1)=\det(c_1^{(\frac{1}{2})}(A))=\frac{1}{2}\det(A)=(\frac{1}{2})(-8)=-4$$

$$A_2=c_{12}(A) \qquad \Rightarrow \det(A_2)=\det(c_{12}(A))=-\det(A)=-(-8)=8$$

$$A_3=c_{23}^{(3)}(A) \quad \Rightarrow \det(A_3)=\det(c_{23}^{(3)}(A))=\det(A)=-8$$

- Thm 3.4: (Conditions that yield a zero determinant)

If A is a square matrix and any of the following conditions is true, then **det (A) = 0**.

(*a*) An entire row (or an entire column) consists of zeros.

(*b*) Two rows (or two columns) are equal.

(*c*) One row (or column) is a multiple of another row (or column).

مباشرة المحددة تنعدم

- Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

- **Note:** You have now surveyed two general methods for evaluating determinants. Of these, the method of using elementary row operations to reduce the matrix to triangular form is usually faster than cofactor expansion along a row or column. If the matrix is large, then the number of arithmetic operations needed for cofactor expansion can become extremely large. For this reason, most computer and calculator algorithms use the method involving elementary row operations. The following table shows the numbers of additions (plus subtractions) and multiplications (plus divisions) needed for each of these two methods for matrices of orders 3, 5, and 10.

	Cofactor Expansion		Row Reduction	
Order *n*	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

In fact, the number of additions alone for the cofactor expansion of an $n \times n$ matrix is $n! - 1$. Because $30! \approx 2.65 \times 10^{32}$, even a relatively small 30×30 matrix would require more than 10^{32} operations. If a computer could do one trillion operations per second, it would still take more than one trillion years to compute the determinant of this matrix using cofactor expansion. Yet, row reduction would take only a few seconds.

Exp

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Cofactor expansion in the 2nd row would be:

$$4 * C_{21} + 5 * C_{22} + 6 * C_{23}$$

C_{21} means the cofactor of row 2, column 1. $So far that's 3 multiplications and 2 additions.

Each cofactor (2x2 matrix) is something like this:

$$A_{22} * A_{11} - A_{21} * A_{12}$$

So that's 1 more addition (really subtraction, same thing) and 2 multiplications.

Since there are 3 cofactors, that means an additional 3 additions and 6 multiplications.

So a total of 9 multiplications and 5 additions.

- Ex 5: (Evaluating a determinant)

$$A=\begin{bmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{bmatrix}$$

مثال لتصفير سطر أو عمود باستعمال عمليات السطر البسيطة ثم من بعد استعمال التوزيع (co-matrices)

Sol:

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{C_{13}^{(2)}}{=}\begin{vmatrix}-3 & 5 & -4\\ 2 & -4 & 3\\ -3 & 0 & 0\end{vmatrix}$$

$$=(-3)(-1)^{3+1}\begin{vmatrix}5 & -4\\ -4 & 3\end{vmatrix}=(-3)(-1)=3$$

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{r_{12}^{(\frac{4}{5})}}{=}\begin{vmatrix}-3 & 5 & 2\\ \frac{-2}{5} & 0 & \frac{3}{5}\\ -3 & 0 & 6\end{vmatrix}$$

$$=(5)(-1)^{1+2}\begin{vmatrix}\frac{-2}{5} & \frac{3}{5}\\ -3 & 6\end{vmatrix}=(-5)(-\frac{3}{5})=3$$

- Ex 6: (Evaluating a determinant)

$$A=\begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

Sol:

$$\det(A)=\begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \underset{\substack{r_{24}^{(1)} \\ r_{25}^{(-1)}}}{=} \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=(1)(-1)^{2+2}\begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{c_{41}^{(-3)}}{=}\begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (1)(-1)^{4+4}\begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \underset{r_{21}^{(1)}}{=} \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix}$$

$$= 5(-1)^{1+3}\begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$

3.3 Properties of Determinants

- Thm 3.5: (Determinant of a matrix product)

$$\det(AB) = \det(A)\det(B)$$

- Notes:

(1) $\det(EA) = \det(E)\det(A)$

(2) $\det(A+B) \neq \det(A)+\det(B)$

(3)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Ex 1: (Check The determinant of a matrix product)

$$A=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} \qquad B=\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A|=\begin{vmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{vmatrix}=-7 \qquad |B|=\begin{vmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{vmatrix}=11$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

- Check:

$$|AB| = |A|\,|B|$$

- Thm 3.6: (Determinant of a scalar multiple of a matrix)

If A is an $n \times n$ matrix and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

- Ex 2:

Given: $A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}$, **and** $\begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5$

Find $|A|$.

Sol:

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- Thm 3.7: (Determinant of an invertible matrix)

A *square* matrix A is invertible (nonsingular) *if and only if*

$\det(A) \neq 0$ شرط القابلية للعكس

- Ex 3: (Classifying square matrices as singular or nonsingular)

$$A=\begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad B=\begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$|A| = 0 \Rightarrow$ A has no inverse (it is singular).

$|B| = -12 \neq 0 \Rightarrow$ B has an inverse (it is nonsingular).

- Thm 3.8: (Determinant of an inverse matrix)

If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

- Thm 3.9: (Determinant of a transpose)

If A is a square matrix, then $\det(A^{T}) = \det(A)$.

- Ex 4:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

(a) $\left|A^{-1}\right| = ?$ (b) $\left|A^{T}\right| = ?$

Sol:

$$\because \quad |A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4$$

$$\therefore \quad \left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}$$

$$\left|A^{T}\right| = |A| = 4$$

- Equivalent conditions for a nonsingular matrix:

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n

(5) A can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

- Ex 5: Which of the following system has **<u>a unique</u>** solution?

$$(a)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$(b)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

من الخاصية السابقة:
يكفي أن تكون مصفوفة المعاملات قابلة للعكس و الذي يعني محددتها غير منعدمة

Sol:

(a) $A\mathbf{x}=\mathbf{b}$

$\because$ $|A|=0$

$\therefore$ This system does not have a unique solution.

(b) $B\mathbf{x}=\mathbf{b}$

$\because$ $|B|=-12\neq 0$

$\therefore$ This system has a unique solution.

3.4 Applications of Determinants

- Matrix of cofactors of A:

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

- Adjoint matrix of A: ***Transpose of Cofactors matrix (Co-matrix)***

$$adj(A) = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

- Thm 3.10: (The inverse of a matrix given by its adjoint)

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

- Application for 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \det(A) = ad - bc$$

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

لاحظ كيف تكتب بالنسبة ل 2x2

- Ex 1 & Ex 2:

$$A=\begin{bmatrix}-1 & 3 & 2\\ 0 & -2 & 1\\ 1 & 0 & -2\end{bmatrix}$$

(a) Find the adjoint of A.

(b) Use the adjoint of A to find A^{-1}

Sol: $\because \quad C_{ij}=(-1)^{i+j}M_{ij}$

$$\Rightarrow C_{11}=+\begin{vmatrix}-2 & 1\\ 0 & -2\end{vmatrix}=4,\quad C_{12}=-\begin{vmatrix}0 & 1\\ 1 & -2\end{vmatrix}=1,\quad C_{13}=+\begin{vmatrix}0 & -2\\ 1 & 0\end{vmatrix}=2$$

$$C_{21}=-\begin{vmatrix}3 & 2\\ 0 & 2\end{vmatrix}=6,\quad C_{22}=+\begin{vmatrix}-1 & 2\\ 1 & 2\end{vmatrix}=0,\quad C_{23}=-\begin{vmatrix}-1 & 3\\ 1 & 0\end{vmatrix}=3$$

$$C_{31}=+\begin{vmatrix}3 & 2\\ -2 & 1\end{vmatrix}=7,\quad C_{32}=-\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=1,\quad C_{33}=+\begin{vmatrix}-1 & 3\\ 0 & -2\end{vmatrix}=2$$

$\Rightarrow$ cofactor matrix of A $\Rightarrow$ adjoint matrix of A

$$[C_{ij}] = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix} \qquad adj(A) = [C_{ij}]^T = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$\Rightarrow$ inverse matrix of A

$$A^{-1} = \frac{1}{\det(A)} adj(A) \qquad \because \det(A) = 3$$

$$= \tfrac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \tfrac{4}{3} & 2 & \tfrac{7}{3} \\ \tfrac{1}{3} & 0 & \tfrac{1}{3} \\ \tfrac{2}{3} & 1 & \tfrac{2}{3} \end{bmatrix}$$

- Check: $AA^{-1} = I$

- Thm 3.11: (**Cramer's Rule or Method**)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

$$A\mathbf{x} = \mathbf{b}$$

$$A = [a_{ij}]_{n\times n} = [A^{(1)}, A^{(2)}, \cdots, A^{(n)}] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

(this system has a unique solution)

$$A_j = \left[A^{(1)}, A^{(2)}, \cdots, A^{(j-1)}, b, A^{(j+1)}, \cdots, A^{(n)}\right]$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \ddots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

(i.e. $\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$)

Solutions $\Rightarrow \boxed{x_j = \dfrac{\det(A_j)}{\det(A)},}\quad j = 1, 2, \cdots, n$

A_j تمثل المصفوفة الأصلية بتعويض العمود الخاص بالمتغير x_j بالعمود b

- Pf:

$$A\,\mathbf{x} = \mathbf{b}, \qquad \det(A) \neq 0$$

$$\Rightarrow \mathbf{x} = A^{-1}\mathbf{b} \quad = \frac{1}{\det(A)} adj(A)\mathbf{b}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} & + & b_2C_{21} & + & \cdots & + & b_nC_{n1} \\ b_1C_{12} & + & b_2C_{22} & + & \cdots & + & b_nC_{n2} \\ & \vdots & & \vdots & & & \\ b_1C_{1n} & + & b_2C_{2n} & + & \cdots & + & b_nC_{nn} \end{bmatrix}$$

$$\Rightarrow x_j = \frac{1}{\det(A)}(b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj})$$

$$= \frac{\det(A_j)}{\det(A)} \qquad j = 1, 2, \cdots, n$$

Note:

Cramer's Rule only works only on **square matrices** that have a non-zero determinant and a unique solution.

- Ex 4: Use **Cramer's rule** to solve the system of linear equations.

$$\begin{aligned} -x &+ 2y - 3z = 1 \\ 2x & + z = 0 \\ 3x &- 4y + 4z = 2 \end{aligned}$$

Sol:

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \qquad \det(A_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8$$

$$\det(A_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15, \qquad \det(A_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5} \qquad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2} \qquad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$