Chapter 1
Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.2 Gaussian Elimination and Gauss-Jordan Elimination Methods

Lecture Outlines:

- Linear Equations and Systems of Linear Equations
- Augmented Matrices
- Elementary Row Operations
- Consistent and Inconsistent Systems
- Solving Syst of Lin Equations using Elementary Row operation
- R.E.F and R.R.E.F properties
- Gaussian Elimination Method
- Gauss-Jordan Elimination Method
- Homogeneous Systems

1.1 Introduction to Systems of Linear Equations

- a linear equation in *n* variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

$a_1, a_2, a_3, \ldots, a_n$, b: real number

a_1: leading coefficient

x_1: leading variable

- Notes:

(1) Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions.

(2) Variables appear only to the first power.

- Ex 1: (Linear or Nonlinear)

Linear $(a)\ 3x+2y=7$

$(b)\ \frac{1}{2}x+y-\pi z=\sqrt{2}$ Linear

Linear $(c)\ x_1-2x_2+10x_3+x_4=0$

$(d)\ (\sin\frac{\pi}{2})x_1-4x_2=e^2$ Linear

Nonlinear $(e)\ xy+z=2$

not the first power

Exponential

$(f)\ e^x-2y=4$ Nonlinear

Nonlinear $(g)\ \sin x_1+2x_2-3x_3=0$

trigonometric functions

$(h)\ \frac{1}{x}+\frac{1}{y}=4$ Nonlinear

not the first power

- a solution of a linear equation in *n* variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

$$x_1 = s_1,\ x_2 = s_2,\ x_3 = s_3,\ \cdots,\ x_n = s_n$$

such that $a_1s_1 + a_2s_2 + a_3s_3 + \cdots + a_ns_n = b$

- Solution set:

the set of <u>all solutions</u> of a linear equation

- Ex 2： (Parametric representation of a solution set)

$$x_1 + 2x_2 = 4$$

a solution: (2, 1), i.e. $x_1 = 2, x_2 = 1$

If you solve for x_1 in terms of x_2, you obtain

$$x_1 = 4 - 2x_2,$$

By letting $x_2 = t$ you can represent the solution set as

$$x_1 = 4 - 2t$$

And the solutions are $\{(4-2t, t) \mid t \in R\}$ or $\{(s, 2-\frac{1}{2}s) \mid s \in R\}$

- a system of **m** linear equations in **n** variables (unknowns):

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\
 & & & & \vdots & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

- Consistent:

  A system of linear equations that has <u>at least one solution</u>.

- Inconsistent:

  A system of linear equations that has <u>no solution</u>.

- Notes:

Every system of linear equations has either

(1) exactly one solution,

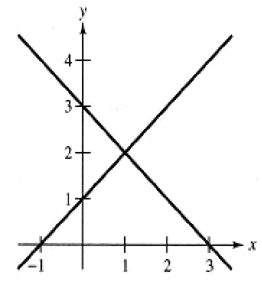
(2) infinitely many solutions, or

(3) no solution.

- Ex 4: (Solutions of a system of linear equations)

(1)
$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$

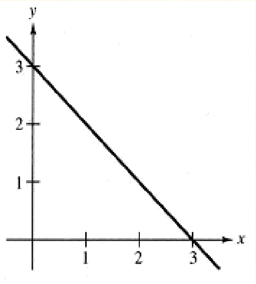
two intersecting lines



exactly one solution

(2)
$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$

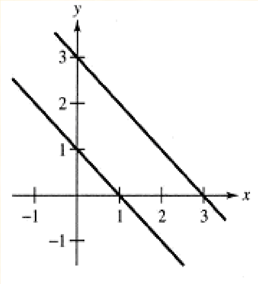
two coincident lines



inifinite number **of solutions**

(3)
$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$

two parallel lines



no solution

- Ex 5: (Using **back substitution** to solve a system in row echelon form)

back substitution: basic idea is to eliminate variables between rows/lines

$$\begin{aligned} x - 2y &= 5 \quad &(1)\\ y &= -2 \quad &(2) \end{aligned}$$

Sol: By substituting $y = -2$ into (1), you obtain

$$\begin{aligned} x - 2(-2) &= 5\\ x &= 1 \end{aligned}$$

The system has exactly one solution: $x = 1,\ y = -2$

- Ex 6: (Using back substitution to solve a system in row echelon form)

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ y + 3z &= 5 \quad (2) \\ z &= 2 \quad (3) \end{aligned}$$

Sol: Substitute $z = 2$ into (2)

$$\begin{aligned} y + 3(2) &= 5 \\ y &= -1 \end{aligned}$$

and substitute $y = -1$ and $z = 2$ into (1)

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 \\ x &= 1 \end{aligned}$$

The system has exactly one solution:

$$x = 1, y = -1, z = 2$$

- Equivalent:

Two systems of linear equations are called **equivalent**

if they have precisely **the same solution set**.

- Notes: Elementary Row Operations method (E.R.O.)

Each of the following operations on a system of linear equations produces **an equivalent system**.

(1) Interchange/swap two equations (rows).

(2) Multiply an equation by a nonzero constant.

(3) Add a multiple of one equation to another equation.

leave the first as it is, and replace the second by the new one

- Ex 7: Solve a system of linear equations (consistent system)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \quad (1) \\ -x & + & 3y & & & = & -4 & \quad (2) \\ 2x & - & 5y & + & 5z & = & 17 & \quad (3) \end{array}$$

Sol: $(1)+(2)\rightarrow(2)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \quad (4) \\ 2x & - & 5y & + & 5z & = & 17 & \end{array}$$

$(1)\times(-2)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \\ & & -y & - & z & = & -1 & \quad (5) \end{array}$$

$(4)+(5)\rightarrow(5)$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \qquad (6) \end{aligned}$$

$(6)\times\frac{1}{2}\rightarrow(6)$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

So the solution is $x=1,\ y=-1,\ z=2$ (only one solution)

Note: all the listed systems are equivalent

- Ex 8: Solve a system of linear equations (inconsistent system)

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \quad (1)\\ 2x_1 - x_2 - 2x_3 &= 2 \quad (2)\\ x_1 + 2x_2 - 3x_3 &= -1 \quad (3) \end{aligned}$$

Sol: $(1)\times(-2)+(2)\rightarrow(2)$

$(1)\times(-1)+(3)\rightarrow(3)$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1\\ 5x_2 - 4x_3 &= 0 \quad (4)\\ 5x_2 - 4x_3 &= -2 \quad (5) \end{aligned}$$

$$(4)\times(-1)+(5)\rightarrow(5)$$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 0 &= -2 \end{aligned}$$

(a false statement)

So the system has no solution (an **inconsistent** system).

- Ex 9: Solve a system of linear equations (infinitely many solutions)

$$\begin{array}{rcrcrcrr} & & x_2 & - & x_3 & = & 0 & \quad (1) \\ x_1 & & & - & 3x_3 & = & -1 & \quad (2) \\ -x_1 & + & 3x_2 & & & = & 1 & \quad (3) \end{array}$$

Sol: $(1) \leftrightarrow (2)$

$$\begin{array}{rcrcrcrr} x_1 & & & - & 3x_3 & = & -1 & \quad (1) \\ & & x_2 & - & x_3 & = & 0 & \quad (2) \\ -x_1 & + & 3x_2 & & & = & 1 & \quad (3) \end{array}$$

$(1) + (3) \rightarrow (3)$

$$\begin{array}{rcrcrcrr} x_1 & & & - & 3x_3 & = & -1 & \\ & & x_2 & - & x_3 & = & 0 & \\ & & 3x_2 & - & 3x_3 & = & 0 & \quad (4) \end{array}$$

$$\begin{aligned} x_1 \quad\quad\quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned}$$

$$\Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

Let $x_3 = t$ No constraints on x_3

then $x_1 = 3t - 1,$

$x_2 = t, \qquad t \in R$

$x_3 = t,$

So this system has infinitely many solutions.

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- $m \times n$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

m rows

n columns

- Notes:

(1) Every entry (or coefficient) a_{ij} in a matrix is a number.

(2) A matrix with m rows/lines and n columns is said to be of size $m \times n$.

(3) If $m = n$, then the matrix is called square of order n.

(4) For a square matrix, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called:

the main diagonal entries.

- Ex 1: Matrix Size

Matrix	Size
$[2]$	1×1
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	2×2
$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$	1×4
$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$	3×2

- Note:

One very common use of matrices is to represent a system of linear equations.

- a system of *m* equations in *n* variables:

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \dots & + & a_{3n}x_n & = & b_3 \\ & & & & \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

Matrix form: $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- Augmented matrix:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ & \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right] = [A \mid b]$$

- Coefficients matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = A$$

- Elementary row operation:

(1) Interchange/swap two rows/lines. $r_{ij} : R_i \leftrightarrow R_j$

(2) Multiply a row by a nonzero constant ***k***. $r_i^{(k)} : (k)R_i \rightarrow R_i$

(3) Add a multiple of a row to another row. $r_{ij}^{(k)} : (k)R_i + R_j \rightarrow R_j$

- Row equivalent:

Two matrices are said to be **row equivalent** if one can be obtained from the other by a finite sequence of **elementary row operation**.

- Ex 2: (Elementary Row Operations E.R.O)

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

- Ex 3: Using elementary row operations to solve a system

Linear System	Associated Augemented Matrix	Elementary Row Operation
$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$	
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$	$r_{12}^{(1)} : (1)R_1 + R_2 \rightarrow R_2$
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$	$r_{13}^{(-2)} : (-2)R_1 + R_3 \rightarrow R_3$

Linear System	Associated Augemented Matrix	Elementary Row Operation
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$	$r_{23}^{(1)} : (1)R_2 + R_3 \rightarrow R_3$
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$	$r_3^{(\frac{1}{2})} : (\frac{1}{2})R_3 \rightarrow R_3$
$\longrightarrow \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$	$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$	**A reduced form of the augmented matrix:** ***System easy to solve***

Definitions of R.E.F. and R.R.E.F.

- Row-echelon form: (conditions 1, 2, 3)
- Reduced row-echelon form: (conditions 1, 2, 3, 4)

(1) A row consisting entirely of zeros occur at the bottom of the matrix.

(2) For each row that does not consist entirely of zeros, the first leftmost nonzero entry is 1 (called **a leading 1**).

(3) For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

(4) Every column that has a leading 1 has zeros in every position above and below its leading 1 (the leftmost leading 1 is the only nonzero entry in the column).

- Ex 4: (Row-echelon form or reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ (row - echelon form)

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row - echelon form)

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ (row - echelon form)

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row - echelon form)

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

- Gaussian elimination method:

  The procedure for reducing a matrix to a row-echelon form.

- Gauss-Jordan elimination method:

  The procedure for reducing a matrix to a reduced row-echelon form.

- Notes:

  (1) Every matrix has an unique reduced row echelon form.

  (2) A row-echelon form of a given matrix is not unique.

  (Different sequences of row operations can produce different row-echelon forms.)

- Ex: (Procedure of Gaussian elimination and Gauss-Jordan elimination)

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix}$$

The first nonzero column

Produce leading 1

$$\xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix}$$

leading 1

Zeros elements below leading 1

Produce leading 1

The first nonzero column

Submatrix

$$\xrightarrow{r_2^{(-\frac{1}{2})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix}$$

leading 1

Zeros elements below leading 1

$$\xrightarrow{r_{23}^{(-5)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 3 & 6 \end{bmatrix}$$

Submatrix

Produce leading 1

$$\xrightarrow{r_3^{(\frac{1}{3})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Zeros elsewhere

leading 1

(row-echelon form)

$$\xrightarrow{r_{31}^{(-6)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(row-echelon form)

$$\xrightarrow{r_{32}^{(4)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(row-echelon form)

$$\xrightarrow{r_{21}^{(3)}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(reduced row-echelon form)

R.E.F and R.R.E.F. STEPS SUMMARY

1- تحديد أقصى عمود الى اليسار والذي ليس كله أصفار

2- تجعل أول عنصر في العمود (أقصى اليسار) يساوي 1 (المتزعم)

(if *a* then multiply by *1/a*)

3- تجعل كل العناصر أسفل الواحد المتزعم تساوي صفر باضافة مضاعفات السطر الأول للأسطر الأخرى

4- نعيد نفس الخطوات التلاث السابقة على المصفوفة بدون السطر الذي يحتوي الواحد المتزعم

5- بتطبيق الخطوات الأربع السابقة سنحصل على صيغة R.E.F.

6- لتحويل المصفوفة الى صورة مختزلة R.R.E.F

نبدأ من اخر سطر غير منعدم و نضيف مضاعفاته الى الأسطر التي فوقه لايجاد أصفار فوق الاحاد المتزعمة

Note:

Steps 1 to 5 called *Gaussian Elimination* Method

Steps 1 to 6 called *Gauss-Jordan Elimination* Method

- Ex 7: Solve the following system using Gauss-Jordan elimination
- method (only one solution)

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Sol:

augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{r_{12}^{(1)},\, r_{13}^{(-2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{r_{23}^{(1)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{r_3^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{32}^{(-3)},\, r_{31}^{(-3)},\, r_{21}^{(2)}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

(row-echelon form) (reduced row-echelon form)

- Ex 8：Solve a system by Gauss-Jordan elimination method (infinitely many solutions)

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \qquad &= 1 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})},\, r_{12}^{(-3)},\, r_2^{(-1)},\, r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

(reduced row - echelon form)

the corresponding system of equations is

$$\begin{aligned} x_1 \qquad + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1 \end{aligned}$$

leading variable : x_1, x_2

free variable : x_3

$$x_1 = 2 - 5x_3$$
$$x_2 = -1 + 3x_3$$

Let $x_3 = t$

$$x_1 = 2 - 5t,$$
$$x_2 = -1 + 3t, \qquad t \in R$$
$$x_3 = t,$$

So this system has infinitely many solutions.

- Homogeneous system of linear equations:

A system of linear equations is said to be homogeneous if all the constant terms are zero.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

- Trivial solution:

$$x_1 = x_2 = x_3 = \cdots = x_n = 0$$

- Nontrivial solution:

  other solutions

- **Notes:**

(0) Any consistent system with m<n has infinite number of solutions

(1) Every homogeneous system of linear equations is consistent.

(2) If the homogenous system has fewer equations than variables (m<n) then it must have an infinite number of solutions.

(3) For a homogeneous system, exactly one of the following is true.

(a) The system has only the trivial solution.

(b) The system has infinitely many nontrivial solutions in addition to the trivial solution.

- Ex 9: Solve the following homogeneous system

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{r_{12}^{(-2)},\, r_2^{(\frac{1}{3})},\, r_{21}^{(1)}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$ (reduced row - echelon form)

leading variable : x_1, x_2

free variable : x_3

Let $x_3 = t$

$x_1 = -2t, x_2 = t, x_3 = t, t \in R$

When $t = 0, x_1 = x_2 = x_3 = 0$ (trivial solution)

Keywords in Section 1.2:

- matrix: مصفوفة
- row: صف
- column: عمود
- entry: عنصر
- size: حجم
- square matrix: مصفوفة مربعة
- symmetric matrix: مصفوفة متماثلة
- trace of a matrix: أثر المصفوفة
- order: ترتيب
- main diagonal: قطر رئيسي
- augmented matrix: مصفوفة موسعة
- coefficient matrix: معامل المصفوفة
- Trivial: بديهي