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A note on some remarkable differential equations on a Riemannian manifold

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A B S T R A C T

The Fischer-Marsden conjecture asserts that an n-dimensional compact manifold admitting a nontrivial solution of the so-called Fischer-Marsden differential equation is necessarily an Einstein space. If this were true, then a classical theorem of Obata would imply that the underlying manifold is either a standard sphere or a Ricci flat space. Although counterexamples to this conjecture have been found by Kobayashi and Lafontaine, it has recently been proved by Cernea and Guan that the Fischer-Marsden conjecture holds, provided that the space of nonconstant solutions of the Fischer-Marsden equation is of dimension at least n, the authors actually proving that in this case (M, g) is nothing but a standard sphere. The main aim of this article is to show that any compact Riemannian manifold of positive Ricci curvature that admits a nontrivial concircular vector field with the potential function satisfying the Fischer-Marsden equation must be isometric to a standard sphere and the converse is also valid. Moreover, we prove that the existence of a nontrivial solution to another remarkable differential equation on Riemannian manifolds, namely the stationary Schrödinger equation, it also leads to a characterization of the sphere, provided that some pinching conditions are satisfied.

1. Introduction

The influence of differential equations and remarkable vector fields in elucidating the geometry of Riemannian manifolds is immense (see, e.g., [1,17,20,30] and the references therein). Some special vector fields such as geodesic, concircular, conformal and Killing vector fields play a key role in obtaining characteriza-

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tions of Euclidean spaces and spheres (cf. [7,13–15,22,39]). In the celebrated works of Obata [32,33], the effect of differential equations on Riemannian manifolds is exhibited, the authors showing that the existence of a nontrivial solution of the equation

$$Hess(f) = -cfg$$

on a complete and connected Riemannian manifold (M, g) of dimension n, where c is a positive constant and $Hess(f)$ is the Hessian of the function f on M, guarantees that (M, g) is isometric to the Euclidean sphere $\mathbf{S}^n(c)$. Similarly, in [19], Fischer and Marsden considered a connected and complete Riemannian space (M, g) admitting a nontrivial solution of the equation

$$(\Delta\rho)\, g + \rho Ric = Hess(\rho), \tag{1}$$

where Δ denotes the Laplace operator that acts on smooth functions ρ on M and Ric stands for the Ricci tensor of g. The above differential equation is known as the Fischer-Marsden equation and ρ is said to be a Fischer-Marsden solution. It is worth noting that this equation naturally appears in general relativity as the static perfect fluid equation (see [28,35,36] and the references indicated therein), being related to the spacelike sections of static spacetimes (see also [16]). Actually, if we consider a Ricci-flat Lorentzian manifold of dimension 4 admitting a timelike Killing vector field and whose orthogonal distribution is integrable, then any integral submanifold of this distribution corresponds to a solution of the Fischer-Marsden equation (1), where g is the induced metric and ρ is the length of the Killing vector field. In [19], the authors demonstrate that if a connected Riemannian space admits a nontrivial solution of the Fischer-Marsden equation, then the scalar curvature $\boldsymbol{\tau}$ is constant. This led Fischer and Marsden to conjecture in [19] that an n-dimensional compact Riemannian manifold admitting a nontrivial solution of (1) must be an Einstein space (for recent developments on this conjecture in contact geometry see [11,34]). If this conjecture were valid, then a result of Obata [31] would imply that the space is Ricci flat or it reduces to a standard n-sphere. However, as proved by Kobayashi [24] and Lafontaine [27], there are many other possibilities, and therefore Fischer and Marsden's conjecture is false. It should be noted that all the counterexamples provided in [24,27] are product manifolds and warped products. It is worth mentioning that the Fischer-Marsden conjecture is closely related to another conjecture by Boucher, Gibbons and Horowitz [6], known as the cosmic no-hair conjecture (for details see [21]).

Afterwards, Cernea and Guan [8] investigated the space $\mathcal{W}$ of the Fischer-Marsden solutions, proving that $\dim \mathcal{W} \leq n+1$ and therefore recovering a result of Corvino [12]. They also showed that any product manifold of the form $\mathbf{S}^m \times N$, for N an Einstein space, provides a Fischer-Marsden solution and thus a new counterexample to the Fischer-Marsden conjecture. Further, Cernea and Guan [9] proved that the Fischer-Marsden conjecture holds if $\dim \mathcal{W} \geq n$, they showing that in this case (M, g) is a standard sphere. In this work we focus on a different setting. We first consider a compact Riemannian space (M, g) having positive Ricci curvature, that admits a nontrivial concircular vector field ξ with potential function ρ satisfying (1) and show that (M, g) must be isometric to the n-dimensional sphere $\mathbf{S}^n(c)$, for a constant $c > 0$. Moreover, we prove that the converse also holds.

In the second part of the article, we deal with another remarkable differential equation on a Riemannian space (M, g) of dimension n, namely

$$\Delta f = -\theta f, \tag{2}$$

where θ is a smooth function. The above equation is nothing but the famous stationary Schrödinger equation (see, e.g., the recent articles [18,25,29,37]), while the solutions of this equation are generalizations of harmonic functions, called L-harmonic functions [26]. We would like to highlight that the Schrödinger equation plays a significant role in quantum mechanics, as it can predict the behavior of dynamical systems, but also in other domains of physics and chemistry, like nonlinear optics, nanomagnetic systems, plasma physics and atomic structure of matter (see, e.g., [2,5,10,23,37], and references therein). The Schrödinger equation on a compact Riemannian manifold (M,g) was investigated in [3,4], the authors emphasizing in particular the influence of the geometry of (M,g) on the behavior of solutions. We are going to show that the stationary Schrödinger equation (2) admits a nontrivial solution f on a compact Riemannian space (M,g), such that some pinching conditions regarding the function θ and Ricci curvature $Ric(\nabla f, \nabla f)$ are satisfied, if and only if the Riemannian space (M,g) is isometric to an n-dimensional sphere $\mathbf{S}^n(c)$.

2. Preliminaries

Let (M,g) be a compact Riemannian manifold of dimension n. Let us denote by ∇ the Riemannian connection and by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M. Then the curvature tensor field R of (M,g) is

$$R(U,V)W = [\nabla_U, \nabla_V]\, W - \nabla_{[U,V]} W, \quad U,V,W \in \mathfrak{X}(M) \tag{3}$$

and the Ricci curvature tensor is

$$Ric(U,V) = \sum_{i=1}^{n} g\left(R(e_i,U)V, e_i\right),$$

where $\{e_1,..,e_n\}$ is a local orthonormal frame on M.

If h is a smooth function on M, then the Hessian operator A_h is a symmetric operator given by

$$A_h U = \nabla_U \nabla h, \quad U \in \mathfrak{X}(M)$$

and the Hessian of h, denoted by $Hess(h)$, is defined as

$$Hess(h)(U,V) = g(A_h U, V), \quad U,V \in \mathfrak{X}(M).$$

Recall next that a vector field $\mathbf{w} \in \mathfrak{X}(M)$ is called concircular if

$$\nabla_U \mathbf{w} = \rho U, \quad U \in \mathfrak{X}(M), \tag{4}$$

where ρ is a smooth function on M called the potential function of $\mathbf{w}$. Moreover, the concircular vector field $\mathbf{w}$ is said to be nontrivial if ρ is a non-constant function. Using equations (3) and (4), we derive

$$R(U,V)\mathbf{w} = U(\rho)V - V(\rho)U, \quad U,V \in \mathfrak{X}(M)$$

and we conclude

$$S\left(\mathbf{w}\right) = -(n-1)\nabla\rho, \tag{5}$$

where S is the Ricci operator defined by

$$g(S(U),V) = Ric(U,V),\ U,V \in \mathfrak{X}(M).$$

Also, on a compact Riemannian space (M,g), the following integral formula holds true (cf. [38])

$$\int_M \left[Ric\,(\mathbf{u},\mathbf{u}) + \frac{1}{2}\left\|\pounds_{\mathbf{u}} g\right\|^2 - \left\|\nabla \mathbf{u}\right\|^2 - (\operatorname{div}\,\mathbf{u})^2 \right] = 0, \tag{6}$$

where $\pounds_{\mathbf{u}} g$ is the Lie derivative of g with respect to the smooth vector field $\mathbf{u}$ on M. For a smooth function h on M, it is easy to see that

$$\pounds_{\nabla h} g = 2Hess(h), \quad \left\|\nabla\nabla h\right\|^2 = \left\|A_h\right\|^2$$

and replacing $\mathbf{u}$ by ∇h in (6), we obtain

$$\int_M \left\|A_h\right\|^2 = \int_M \left((\Delta h)^2 - Ric\,(\nabla h, \nabla h) \right). \tag{7}$$

3. Characterizations of spheres

We first suppose in this section that the potential function ρ of a nontrivial concircular vector field $\mathbf{w}$ on an n-dimensional Riemannian space (M,g) satisfies the Fischer-Marsden equation (1) and show that if (M,g) has positive Ricci curvature, then it is isometric to an n-dimensional sphere $\mathbf{S}^n(c)$ and the converse also holds. As $\mathbf{S}^n(c)$ is an Einstein space, it follows in particular that the Fischer-Marsden conjecture is true in the above setting. Indeed, we prove the following:

Theorem 3.1. *Let (M,g) be an n-dimensional connected and compact Riemannian space of positive Ricci curvature. Then (M,g) admits a nontrivial concircular vector field $\mathbf{w}$ with potential function ρ satisfying the Fischer-Marsden equation (1) if and only if the Riemannian space (M,g) is isometric to the n-dimensional sphere $\mathbf{S}^n(c)$, for a constant $c > 0$.*

Proof. Suppose first $\mathbf{w}$ is a nontrivial concircular vector field such that the potential function ρ of $\mathbf{w}$ satisfies the Fischer-Marsden equation. Then the scalar curvature $\boldsymbol{\tau}$ is a constant (cf. [19]) and taking trace in equation (1) we obtain

$$\Delta\rho = -\frac{\boldsymbol{\tau}}{n-1}\rho. \tag{8}$$

Multiplying the above equation by ρ and integrating by parts, we derive

$$\int_M \left\|\nabla\rho\right\|^2 = \frac{\boldsymbol{\tau}}{n-1}\int_M \rho^2. \tag{9}$$

As $\mathbf{w}$ is a nontrivial concircular vector field, the potential function ρ is a non-constant function and thus the above equation implies that the constant $\boldsymbol{\tau} > 0$. We put $\boldsymbol{\tau} = n(n-1)c$ and we have a constant $c > 0$. Now, using equation (4), we have div $\mathbf{w} = n\rho$ and the equation (5) gives

$$\begin{aligned} Ric(\mathbf{w},\mathbf{w}) &= -(n-1)\mathbf{w}\,(\rho) \\ &= -(n-1)\left(\operatorname{div}(\rho\mathbf{w}) - n\rho^2\right). \end{aligned}$$

Integrating the last equation, we obtain

$$\int_M Ric(\mathbf{w},\mathbf{w}) = n(n-1)\int_M \rho^2. \tag{10}$$

Also, equation (5) implies

$$Ric(\nabla\rho, \mathbf{w}) = -(n-1)\left\|\nabla\rho\right\|^2. \tag{11}$$

Next, we have

$$Ric\left(\nabla\rho + c\mathbf{w}, \nabla\rho + c\mathbf{w}\right) = Ric\left(\nabla\rho, \nabla\rho\right) + 2cRic(\nabla\rho, \mathbf{w}) + c^2 Ric(\mathbf{w}, \mathbf{w})$$

and integrating the previous equation, while using (10) and (11), we infer

$$\begin{aligned}\int_M Ric\left(\nabla\rho + c\mathbf{w}, \nabla\rho + c\mathbf{w}\right) = &\int_M Ric\left(\nabla\rho, \nabla\rho\right)\\ &- \int_M \left(2(n-1)c\left\|\nabla\rho\right\|^2 - n(n-1)c^2\rho^2\right).\end{aligned}$$

Using (7) and (9) in the last expression, we derive

$$\int_M Ric\left(\nabla\rho + c\mathbf{w}, \nabla\rho + c\mathbf{w}\right) = \int_M \left((\Delta\rho)^2 - \left\|A_\rho\right\|^2 - 2\boldsymbol{\tau}c\rho^2 + n(n-1)c^2\rho^2\right).$$

Now, using $\boldsymbol{\tau} = n(n-1)c$ and equation (8) in the preceding equation, we obtain

$$\int_M Ric\left(\nabla\rho + c\mathbf{w}, \nabla\rho + c\mathbf{w}\right) = \int_M \left(nc^2\rho^2 - \left\|A_\rho\right\|^2\right). \tag{12}$$

Also, as ρ is a solution of the Fischer-Marsden equation (1), using equation (8) we derive

$$A_\rho = \rho\left(S - ncI\right),$$

which gives

$$\begin{aligned}\left\|A_\rho\right\|^2 &= \rho^2\left(\left\|S\right\|^2 + n^3c^2 - 2nc\boldsymbol{\tau}\right)\\ &= \rho^2\left[\left\|S\right\|^2 - \frac{\boldsymbol{\tau}^2}{n} + n(n-1)^2c^2 + n^3c^2 - 2nc\boldsymbol{\tau}\right].\end{aligned}$$

Inserting $\boldsymbol{\tau} = n(n-1)c$ in the last term, we get

$$\left\|A_\rho\right\|^2 = \rho^2\left(\left\|S\right\|^2 - \frac{\boldsymbol{\tau}^2}{n} + nc^2\right).$$

Using now the last equation in (12), we arrive at

$$\int_M Ric\left(\nabla\rho + c\mathbf{w}, \nabla\rho + c\mathbf{w}\right) = \int_M \rho^2\left(\frac{\boldsymbol{\tau}^2}{n} - \left\|S\right\|^2\right)$$

and applying the Schwarz's inequality $\left\|S\right\|^2 \geq \frac{\boldsymbol{\tau}^2}{n}$, we derive

$$\int_M Ric\left(\nabla\rho + c\mathbf{w}, \nabla\rho + c\mathbf{w}\right) \leq 0.$$

But the Ricci curvature is positive and thus we get

$$\nabla\rho + c\mathbf{w} = 0.$$

Taking the covariant derivative in the last expression and making use of equation (4), we get

$$\nabla_U \nabla\rho = -c\rho U, \quad U \in \mathfrak{X}(M)$$

and as the function ρ is non-constant, we conclude that (M, g) is isometric to the n-dimensional sphere $\mathbf{S}^n(c)$.

Conversely, let us suppose that (M, g) is isometric to $\mathbf{S}^n(c)$. Then treating $\mathbf{S}^n(c)$ as a hypersurface of the Euclidean space $\mathbf{E}^n$, we have the concircular vector field $\mathbf{w}$ on $\mathbf{S}^n(c)$ given by the tangential projection of a nonzero constant vector field $\overrightarrow{\mathbf{a}}$ on the Euclidean space $\mathbf{E}^n$. It satisfies

$$\nabla_U \mathbf{w} = -\sqrt{c}\rho U, \quad \nabla\rho = \sqrt{c}\mathbf{w}, \quad U \in \mathfrak{X}\left(\mathbf{S}^n(c)\right), \tag{13}$$

where $\rho = \langle \overrightarrow{\mathbf{a}}, N\rangle$, N being the unit normal to the sphere $\mathbf{S}^n(c)$ in the Euclidean space $\mathbf{E}^n$ and $\langle\ ,\ \rangle$ is the Euclidean metric. The equations in (13) imply that the vector field $\mathbf{w}$ on $\mathbf{S}^n(c)$ is concircular with potential function $-\sqrt{c}\rho$ and

$$\Delta\rho = -nc\rho.$$

Also, using (13) it follows

$$A_\rho = -c\rho I,$$

that is

$$Hess(\rho) = -c\rho g.$$

But as the Ricci tensor on the sphere $\mathbf{S}^n(c)$ is given by

$$Ric = (n-1)cg,$$

we derive

$$(\Delta\rho)\, g + \rho Ric = -nc\rho g + (n-1)c\rho g = -c\rho g = Hess(\rho).$$

Hence the function ρ satisfies the Fischer-Marsden equation (1) and consequently the potential function $-\sqrt{c}\rho$ also satisfies equation (1). $\square$

In the next part of this section, we are interested in showing that the existence of a nontrivial solution of the stationary Schrödinger equation (2) also leads to a characterization of the n-dimensional sphere $\mathbf{S}^n(c)$, provided that some pinching conditions are satisfied. Indeed, we prove the following.

Theorem 3.2. *Let (M, g) be a compact Riemannian space of dimension n. Then (M, g) admits a nontrivial solution of the equation $\Delta h = -\theta h$ such that the smooth function θ satisfies $c < \theta \leq nc$, for a constant $c > 0$, and the Ricci curvature $Ric(\nabla h, \nabla h)$ in the direction of ∇h is bounded below by $(n-1)c$, if and only if the Riemannian space (M, g) is isometric to the n-dimensional sphere $\mathbf{S}^n(c)$.*

Proof. Suppose h is a non-constant function on M that satisfies

$$\Delta h = -\theta h \tag{14}$$

and $c < \theta \leq nc$ holds for a constant $c > 0$. Multiplying equation (14) by h and integrating by parts, we obtain

$$\int_M \|\nabla h\|^2 = \int_M \theta h^2. \tag{15}$$

Now, choosing a local orthonormal frame $\{e_1, ..., e_n\}$, we have

$$\begin{aligned}\|A_h + chI\|^2 &= \sum_{i=1}^{n} g\left(A_h e_i + che_i, A_h e_i + che_i\right) \\ &= \|A_h\|^2 + nc^2h^2 + 2ch\Delta h.\end{aligned}$$

Integrating the above expression and making use of (14), we derive

$$\int_M \|A_h + chI\|^2 = \int_M \left[\|A_h\|^2 + c\,(nc - 2\theta)\,h^2\right]$$

and taking account of (7), the last equation implies

$$\int_M \|A_h + chI\|^2 = \int_M \left[(\Delta h)^2 + c\,(nc - 2\theta)\,h^2 - Ric\,(\nabla h, \nabla h)\right]. \tag{16}$$

Now, using the bound for $Ric(\nabla h, \nabla h)$, namely

$$Ric\,(\nabla h, \nabla h) \geq (n-1)c\,\|\nabla h\|^2,$$

and equation (14) in (16), we arrive at

$$\int_M \|A_h + chI\|^2 \leq \int_M \left[\left(\theta^2 + c\,(nc - 2\theta)\right) h^2 - (n-1)c\,\|\nabla h\|^2\right].$$

Using equation (15) in the above inequality, we conclude

$$\int_M \|A_h + chI\|^2 \leq \int_M (\theta - nc)\,(\theta - c)\,h^2.$$

Note that the pinching condition $c < \theta \leq nc$ implies that the integrand on the right hand side of the inequality is non-positive and thus we conclude

$$\int_M \|A_h + chI\|^2 \leq 0.$$

Therefore we derive that

$$A_h + chI = 0$$

and we have

$$\nabla_U \nabla h = -chU, \quad U \in \mathfrak{X}(M).$$

But the above equation is nothing but Obata's differential equation for the non-constant function h and real constant $c > 0$ (cf. [32,33]). Therefore, we deduce that (M, g) is isometric to the n-dimensional sphere $\mathbf{S}^n(c)$.

The converse statement is trivial due to the fact that on $\mathbf{S}^n(c)$ we have the eigenfunction h corresponding to first nonzero eigenvalue nc, i.e. $\Delta h = -nch$. □

Remark 3.3. It is worth mentioning that there are several examples of noncompact Riemannian manifolds admitting nontrivial solutions of the stationary Schrödinger equation (2). Let us illustrate two such examples in the following.

(i) Consider the open subset M of the Euclidean space $\mathbf{E}^n$ given by

$$M = \left\{u \in \mathbf{E}^n : n\sqrt{c-1} < \|u\| < n\sqrt{nc-1}\right\},$$

where $c > 1$ is a real constant. Let g be the Euclidean metric on the Euclidean space $\mathbf{E}^n$. Now, on the n-dimensional Riemannian manifold (M, g), we consider the smooth function f defined by

$$f = e^{-\frac{1}{2n}\|u\|^2}.$$

Then it follows that the gradient ∇f of f is

$$\nabla f = -\frac{1}{n} f \left(u_1 \frac{\partial}{\partial u_1} + ... + u_n \frac{\partial}{\partial u_n}\right),$$

where $u_1, ..., u_n$ are the Euclidean coordinates and the Laplace operator Δ acting on f satisfies

$$\Delta f = -\theta f,$$

where the smooth function θ is given by

$$\theta = \frac{1}{n^2}\|u\|^2 + 1.$$

Thus, f is a nontrivial solution of the stationary Schrödinger equation (2) on the Riemannian manifold (M, g) and it satisfies $c < \theta < nc$.

(ii) Consider the open subset $M \subset \mathbf{E}^n$, $n > 3$, defined by

$$M = \left\{u \in \mathbf{E}^n : \sqrt{\frac{n-3}{nc}} < \|u\| < \sqrt{\frac{n-3}{c}}\right\},$$

where $c > 0$ is a real constant. Define the smooth function f on M by

$$f = -\frac{1}{n-3}\|u\|^{3-n}.$$

Then we get

$$\nabla f = \|u\|^{-(n-1)} \left(u_1 \frac{\partial}{\partial u_1} + ... + u_n \frac{\partial}{\partial u_n}\right)$$

and

$$\Delta f = \frac{3-n}{\|u\|^2} f.$$

Thus, f is a nontrivial solution of the stationary Schrödinger equation $\Delta f = -\theta f$, where the smooth function θ is given by

$$\theta = \frac{n-3}{\|u\|^2}$$

and it satisfies $c < \theta < nc$.

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