

King Saud University
Department Of Mathematics.
M-203 [Final Examination](Differential and Integral Calculus)
(I-Semester 1432/1433)

Max. Marks: 50

Time: 3 hrs

Marking Scheme: Q.No:1[3+4+5]; Q.No:2[4+4+4]; Q.No:3[4+4+4]; Q.No:4[4+5+5]

- Q. No: 1** (a) Discuss the convergence of the sequence $\{\ln(n+1) - (\ln n)\}$, and find the limit if it exists.
- (b) Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{2n^2 - 1}$ is absolutely convergent, conditionally convergent, or divergent.
- (c) Find the interval of convergence and radius of convergence of the power series $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2} (x+2)^n$.
- Q. No: 2** (a) Use the **Maclaurin** series for $f(x) = \sin x$ to approximate $\int_0^1 x \sin(2x) dx$ (using three non-zero terms).
- (b) Evaluate the integral $\int_0^1 \int_{\sqrt{x}}^1 e^{-y^3} dy dx$.
- (c) Find the surface area of the surface bounded by the graphs of $x^2 + y^2 - 2 = z$, $z \leq 0$.
- Q. No: 3** (a) Find the Volume of the region bounded by $z=0$ and the paraboloid $z = 16 - (x^2 + y^2)$ using triple integral.
- (b) Use spherical coordinates to evaluate the integral $\iiint_Q (x^2 + y^2 + z^2)^{-3/2} dV$, where Q is a solid region bounded between the two spheres $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 + z^2 = 4$ in the first octant.
- (c) Find the work done by the force $\vec{F} = y\vec{i} - x\vec{j}$ in moving a particle from (1,1) to (-1,1) along the curve $x^2 = y$.
- Q. No: 4** (a) Use Green's theorem to evaluate the line integral $\oint_C xy dx + x dy$, where C is the boundary of the region bounded by the graphs of the equations $y^2 = x$ and $x^2 = y$.
- (b) Use the divergence theorem to find the flux of the vector field $\vec{F}(x, y, z) = (x + \sin y)\vec{i} - (y + \cos z)\vec{j} + (2z - e^{-x})\vec{k}$ through the surface S, which is the portion of the cone $z = \sqrt{x^2 + y^2}$ bounded by the plane $z = 1$.
- (c) Use Stokes's theorem to evaluate the integral $\iint_S (\text{curl } \vec{F}) \cdot \vec{n} dS$, where $\vec{F}(x, y, z) = -y\vec{i} + z\vec{j} + x\vec{k}$, S is the paraboloid $z = 2 - (x^2 + y^2)$ cut off by the plane $z = 1$, and \vec{n} is the unit upper normal vector to the surface S.



Q1 a

$$\ln(n+1) - \ln(n)$$

$$\Rightarrow \ln\left(\frac{n+1}{n}\right)$$

$$\lim_{n \rightarrow \infty} \ln\left(\frac{n+1}{n}\right)$$

$$\Rightarrow \ln(1) = 0 \quad \therefore \text{c/g}$$

$$\text{Q2} \sum_{n=1}^{\infty} (-1)^n \frac{2n}{2n^2-1}$$

by using L.B.C

$$b_n = \frac{2n}{2n^2} = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{2n}{2n^2-1} \cdot \frac{n}{2}$$

$$\lim_{n \rightarrow \infty} \frac{2n^2}{4n^2-2} = \frac{1}{2} > 0$$

$b_n = \frac{1}{n}$ p-series $\rightarrow p=1$ harmonic series ~~div~~ div

by using A.S.T $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{2n^2-1}$

$$f(n) = \frac{2(2n-1) - 2n(4n)}{(2n^2+1)^2} = \frac{4n-1-8n^2}{(2n^2+1)^2} < 0 \quad \therefore \text{decreasing}$$

$$\lim_{n \rightarrow \infty} \frac{2n}{2n^2-1} \xrightarrow{\frac{\infty}{\infty}} \frac{2}{2n-\frac{1}{n}} \xrightarrow{\frac{\infty}{\infty}} \frac{2}{\infty} = 0 \quad \therefore \text{c/g}$$

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{2n}{2n^2-1}$ is c/c
conditionally convergent



$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2} (x+2)^n$$

$$U_n = (-1)^n \frac{2^n}{n^2} (x+2)^n$$

$$U_{n+1} = (-1)^{n+1} \frac{2^{n+1}}{(n+1)^2} (x+2)^{n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{U_{n+1}}{U_n} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 2^{n+1} (x+2)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^n 2^n (x+2)^n} \right|$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{2n^2}{(n+1)^2} \right| |x+2|$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{2n^2}{n^2 + 2n + 1} \right| |x+2|$$

$$-1 < 2|x+2| < 1$$

$$-\frac{1}{2} < x+2 < \frac{1}{2}$$

$$-\frac{1}{2} - 2 < x < \frac{1}{2} - 2$$

$$-\frac{5}{2} < x < -\frac{3}{2}$$

at $x = -\frac{5}{2}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2} (x+2)^n$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2} \left(-\frac{5}{2} + 2\right)^n$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2} \left(-\frac{1}{2}\right)^n$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2} (-1)^n \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad p\text{-series} \Rightarrow p=2$$

∴ c.g

at $x = -\frac{3}{2}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2} (x+2)^n$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2} \left(\frac{1}{2}\right)^n$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2} \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \quad \text{by A.S.T}$$

$$f(n) = \frac{-2}{n^3} < 0 \quad \therefore \text{decreasing}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = \frac{1}{\infty} = 0$$

∴ c.g

interval $\left[-\frac{5}{2}, -\frac{3}{2}\right]$

$$\text{radius} = \frac{-\frac{3}{2} + \frac{5}{2}}{2} = \frac{1}{2}$$



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Q.2 (a)

$$\begin{aligned}
 f(x) &= \sin x & f(0) &= 0 \\
 f'(x) &= \cos x & f'(0) &= 1 \\
 f''(x) &= -\sin x & f''(0) &= 0 \\
 f'''(x) &= -\cos x & f'''(0) &= -1 \\
 f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\
 f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{M.S. } f(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots \\
 &= \cancel{0} + \cancel{x} + \dots \\
 &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!}
 \end{aligned}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

$$\sin 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

$$x \sin 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+2}}{(2n+1)!} = 2x^2 - \frac{2x^4}{3} + \frac{2x^6}{5} - \dots$$

$$x \sin 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+2}}{(2n+1)!}$$

$$\begin{aligned}
 \int_0^1 x \sin 2x &= \int_0^1 \left(2x^2 - \frac{2x^4}{3} + \frac{2x^6}{5} - \dots \right) dx \\
 &= \left. \frac{2}{3}x^3 - \frac{2}{15}x^5 + \frac{2}{35}x^7 - \dots \right|_0^1 \\
 &= \frac{2}{3} - \frac{2}{15} + \frac{2}{35} - \dots = \frac{62}{105} \approx 0.59
 \end{aligned}$$

$$\begin{aligned}
 x \sin 2x &= 2x^2 - \frac{2x^4}{3!} + \frac{2x^6}{5!} - \dots \\
 \int_0^1 x \sin 2x &= \left. 2x^2 - \frac{2x^4}{6} + \frac{2x^6}{120} - \dots \right|_0^1 \\
 &= \frac{2}{3} - \frac{2}{30} + \frac{2}{240} - \dots = \frac{62}{105}
 \end{aligned}$$

$$\int_0^1 x \sin 2x = \frac{2}{3} - \frac{2}{15} + \frac{2}{35} = \frac{62}{105} \approx 0.59$$

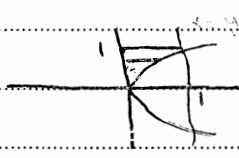
$$\int_0^1 x \sin 2x = \frac{2x^3}{3} - \frac{2x^5}{30} + \frac{2x^7}{240} \Big|_0^1 = \frac{2}{3} - \frac{2}{30} + \frac{2}{240}$$

Q.2 (b)

$$\int_0^1 \int_{\sqrt{x}}^1 e^{-y^3} dy dx$$

$$= \frac{2}{3} - \frac{2}{30} + \frac{2}{240}$$

$x=0 \rightarrow 1$
 $y=\sqrt{x} \quad y=1$



$$\int_0^1 \int_0^y e^{-x^3} dx dy$$

$$\int_0^1 x e^{-x^3} \Big|_0^y dy \Rightarrow \int_0^1 y^2 e^{-y^3} dy$$

$$\Rightarrow \left. -\frac{1}{3} e^{-y^3} \right|_0^1 = \frac{2}{3}$$



22 (c)

$$\iint \sqrt{x^2 + y^2 + 1} \, dA$$

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} \, r \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^{\sqrt{2}} (4r^2 + 1)^{\frac{1}{2}} \, r \, dr \, d\theta$$

$$\frac{1}{8} \int_0^{2\pi} \left. \frac{4r^2 + 1}{\frac{3}{2}} \right|_0^{\sqrt{2}} d\theta$$

$$\frac{1}{8} \cdot \frac{2}{3} \int_0^{2\pi} (4)^{\frac{3}{2}} - (1)^{\frac{3}{2}} d\theta$$

$$\frac{1}{12} \cdot 2\pi \int_0^{2\pi} d\theta$$

$$\frac{13}{6} \pi$$

$$= \frac{13}{6} 2\pi = \frac{13}{3} \pi$$

3 (a)

$$V = \iiint f(x, y) \, dV$$

$$\int_0^{2\pi} \int_0^4 \int_0^4 (16 - r^2) \, r \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^4 (4r - r^3) \, dr \, d\theta$$

$$\int_0^{2\pi} \left. \left(2r^2 - \frac{r^4}{4} \right) \right|_0^4 d\theta$$

$$\int_0^{2\pi} 128 - 64 \, d\theta$$

$$64 \int_0^{2\pi} d\theta$$

$$64 \cdot 2\pi$$

$$z = x^2 + y^2 - 2 \quad z \leq 0$$

$$f_x = 2x \rightarrow f_x^2 = 4x^2$$

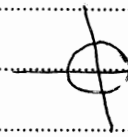
$$f_y = 2y \rightarrow f_y^2 = 4y^2$$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$$

$$x^2 + y^2 = 2$$

$$r: 0 \rightarrow \sqrt{2}$$

$$\theta: 0 \rightarrow 2\pi$$



$$z = 16 - x^2 - y^2$$

$$x^2 + y^2 = 16$$

$$r^2 = 16$$

$$r = 4$$

$$r: 0 \rightarrow 4$$

$$\theta: 0 \rightarrow 2\pi$$



$$\rho^2 \frac{1}{\rho^3} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\frac{2x}{x^2} \quad 2x \cdot x^{-1} \quad 2x \cdot x^{-2} \quad -2x^{-1} \quad -\frac{2}{x}$$



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Q 3 b

$$\iiint \rho \frac{1}{\rho^3} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \frac{1}{\rho^3} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \frac{1}{\rho} \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{\pi/2} \int_0^{\pi/2} \left. \frac{1}{\rho} \right|_1^2 \sin \phi \, d\phi \, d\theta$$

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2} \sin \phi \, d\phi \, d\theta$$

$$-\frac{1}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi \, d\phi \, d\theta$$

$$-\frac{1}{4} \int_0^{\pi/2} -\cos \phi \Big|_0^{\pi/2} d\theta$$

$$\frac{1}{4} \int_0^{\pi/2} (0 - 1) d\theta$$

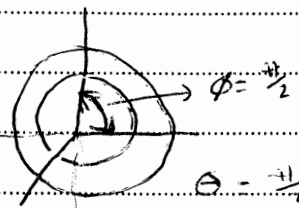
$$-\frac{1}{4} \int_0^{\pi/2} d\theta$$

$$-\frac{1}{4} \theta \Big|_0^{\pi/2}$$

$$-\frac{1}{4} \cdot \frac{\pi}{2} = -\frac{\pi}{8}$$

$$x^2 + y^2 + z^2 = 1 \quad \rho = 1$$

$$x^2 + y^2 + z^2 = 4 \quad \rho = 2$$



$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{\pi/2} \int_0^{\pi/2} \rho^2 \Big|_1^2 \sin \phi \, d\phi \, d\theta$$

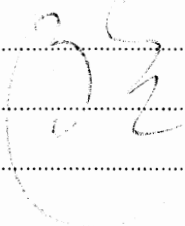
$$\int_0^{\pi/2} \int_0^{\pi/2} 4 - 1 \sin \phi \, d\phi \, d\theta$$

$$3 \int_0^{\pi/2} \cos \phi \Big|_0^{\pi/2} d\theta$$

$$3 \int_0^{\pi/2} 0 - 1 d\theta$$

$$3 (\theta) \Big|_0^{\pi/2}$$

$$3 \cdot \frac{\pi}{2} = \frac{3\pi}{2}$$





Q 3 C

$$W = \int_C f \cdot dr$$

$$f \cdot dr = x dx - x dy$$

Let

$$\begin{aligned} \text{Let } x &= t, & y &= t^2 \\ x' &= 1, & y' &= 2t \end{aligned}$$

$$\int_{-1}^1 t^2(t) - t(2t) dt$$

$$2 \int_0^1 t^2 - 2t^2 dt$$

$$2 \int_0^1 -t^2 dt$$

$$-2 \int_0^1 t^2 dt$$

$$-2 \left(\frac{t^3}{3} \right) \Big|_0^1$$

$$-2 \left(\frac{1}{3} \right) = -\frac{2}{3}$$

$$\Rightarrow W = -\frac{2}{3}$$

C · dr

~~$$\int_C x dx - x dy$$~~

$$\int_C f \cdot dr = x dx - x dy$$

$$\text{Let } x = t, \quad y = t^2$$

$$x' = 1, \quad y' = 2t$$

$$\left(\frac{1}{-1} \right) \left(\frac{1}{-1} \right)$$

36
27



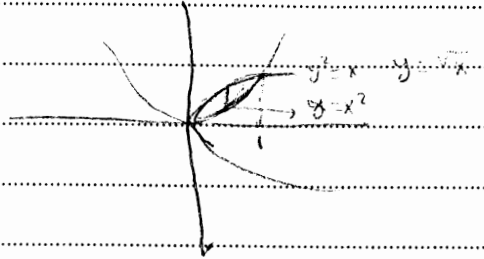
Q4 a

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$M = xy \Rightarrow \frac{\partial M}{\partial y} = x$$

$$N = x \Rightarrow \frac{\partial N}{\partial x} = 1$$

$$y^2 = x \quad y = x^2$$



$$(x^2 - x) = 0$$

$$x(x-1) = 0$$

$$\int_0^1 \int_{x^2}^{x^2} (1-x) dy dx$$

$$\int_0^1 y \cdot xy \Big|_{x^2}^{x^2} dx$$

$$(x^2 - x^3) - (x\sqrt{x} - x\sqrt{x})$$

$$\int_0^1 x^2 - x^3 - x^{3/2} - x^{3/2} dx$$

$$\frac{x^3}{3} + \frac{x^4}{4} - \frac{2}{5} x^{5/2} + \frac{2}{5} x^{5/2} \Big|_0^1$$

$$\frac{1}{3} + \frac{1}{4} - \frac{2}{3} + \frac{2}{5}$$

$$= \frac{19}{60} = 0.317$$

$$\int_0^1 \int_{x^2}^{x^2} (1-x) dy dx$$

$$\int_0^1 y \cdot xy \Big|_{x^2}^{x^2} dx$$

$$\int_0^1 (x\sqrt{x} - x\sqrt{x}) - (x^2 - x^3) dx$$

$$\int_0^1 x^{3/2} - x^{3/2} - x^2 + x^3 dx$$

$$\frac{2}{5} x^{5/2} - \frac{2}{5} x^{5/2} - \frac{x^3}{3} + \frac{x^4}{4} \Big|_0^1$$

$$\frac{2}{3} - \frac{2}{3} - \frac{1}{3} + \frac{1}{4}$$

$$\frac{11}{60} = 0.183$$

$$= \frac{11}{60} = 0.183$$

Q4 [b]

$$\int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \mathbf{F} \, dV$$

$$z = \sqrt{x^2 + y^2} \quad z=1$$

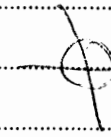
$$\mathbf{F}(x, y, z) = (x + \sin y)\mathbf{i} + (y + \cos z)\mathbf{j} + (2z - e^x)\mathbf{k}$$

$$z^2 = x^2 + y^2$$

$$x^2 + y^2 = 1$$

$$r^2 = 1$$

$$r = 1$$



$$r \, d\theta \rightarrow 1 \, d\theta$$

$$0 \leq \theta \leq 2\pi$$

$$\nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= 1 + 1 + 2$$

$$= 4$$

$$4 \int_0^{2\pi} \int_0^1 \int_0^1 4 \, r \, dz \, dr \, d\theta$$

$$2 \int_0^{2\pi} \int_0^1 r z \, dr \, d\theta$$

$$2 \int_0^{2\pi} \int_0^1 r^2 - r \, dr \, d\theta$$

$$2 \int_0^{2\pi} \left[\frac{r^3}{3} - \frac{r^2}{2} \right]_0^1 d\theta$$

$$2 \int_0^{2\pi} \left[\frac{1}{3} - \frac{1}{2} \right] d\theta$$

$$-\frac{1}{3} \int_0^{2\pi} d\theta$$

$$-\frac{1}{3} (\theta)_0^{2\pi}$$

$$-\frac{1}{3} (2\pi) = -\frac{2\pi}{3}$$



Q 4 (c)

$$\iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, ds = \oint_C \vec{F} \cdot d\vec{r}$$

$$\text{L.H.S.} = \oint_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = -y\vec{i} + z\vec{j} + x\vec{k}$$

$$\int_C d\vec{r} = -x \, dx + z \, dy + x \, dz$$

$$= -y \, dx + dy$$

$$\begin{cases} z=1 \\ dz=0 \end{cases}$$

$$M = -y \quad \frac{\partial M}{\partial y} = -1$$

$$N = 1 \quad \frac{\partial N}{\partial x} = 0$$

$$\int_0^{2\pi} \int_0^1 0 \, r \, dr \, d\theta = 0$$

$$= 0$$

$$\therefore \iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, ds = \oint_C \vec{F} \cdot d\vec{r} = 0$$

$$M = -y \quad \frac{\partial M}{\partial y} = -1$$

$$N = 1 \quad \frac{\partial N}{\partial x} = 0$$

$$\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dA$$

$$\int_0^{2\pi} \int_0^1 -r \, dr \, d\theta$$

$$\int_0^{2\pi} \left[-\frac{r^2}{2} \right]_0^1 \, d\theta$$

$$-\frac{1}{2} \int_0^{2\pi} d\theta$$

$$-\frac{1}{2} (\theta)_0^{2\pi}$$

$$\vec{F} = -y\vec{i} + z\vec{j} + x\vec{k}$$

$$= -y \, dx + z \, dy + x \, dz$$

$$z=1 \quad dz=0$$

$$= -y \, dx + dy$$

$$x^2 + y^2 = 2$$

$$x^2 + y^2 = 1$$

$$r=1$$

$$r=1$$

$$r=0 \rightarrow 1$$

$$\theta: 0 \rightarrow 2\pi$$

