Department of Mathematics
King Saud University
Final Examination
M-203(Differential & Integral Calculus)
First Semester (1434/1435)

Max. Marks: 40 **Time**: 180 minutes

Marking Scheme:	**Q.No:1**[3+3+3+3],	**Q.No:2**[3+3+3+3],	**Q.No:3**[3+4+6+3]

Q. No: 1 (a) Find the sum of the series $\sum_{n=1}^{\infty}\left[\frac{1}{2^n}+\frac{1}{e^{n-1}}\right]$.

(b) Test the convergence or divergence of the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^5+1}}$.

(c) Find the radius of convergence and interval of convergence for the power series

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n\,4^n}.$$

(d) Give Maclaurin Series of $f(x)=\sin x$ and use its first three non-zero terms to approximate the integral $\int_{-1}^{1}\frac{\sin x}{x}dx$.

Q. No: 2 (a) Evaluate the integral $\int_0^2\int_{y/2}^{1}e^{x^2}dxdy$.

(b) Find the surface area of portion of the sphere $x^2+y^2+z^2=1$ that lies inside the cylinder $x^2+y^2=y$.

(c) Evaluate the integral $\iiint_Q xdV$ where Q is the solid region bounded by the coordinate planes and the plane $x+y+z=1$.

(d) Use **cylindrical coordinates** to evaluate the integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4}(x^2+y^2)dzdydx$.

Q. No: 3 (a) If $\vec{F}(x,y)=\frac{y}{x}\vec{i}+\vec{j}$, show that $\int_C\vec{F}\cdot d\vec{r}$ is not independent of path and evaluate $\int_C\vec{F}\cdot d\vec{r}$, where C is given by $y=\ln x$, with $x\in[1,e]$.

(b) Use **Green's theorem to evaluate the line integral** $\oint_C(x-y)dx+xdy$, where C is the boundary of the region bounded by $x^2+y^2=9$ with $0\le\theta\le\frac{\pi}{9}$.

(c) Verify the **Divergence theorem** by evaluating both the surface integral and the triple integral, for the vector field $\vec{F}(x,y,z)=x\vec{i}+y\vec{j}+z\vec{k}$ and surface S, where S is the upper half of the sphere $x^2+y^2+z^2=9$.

(d) Use Stokes's theorem to evaluate $\oint\vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y,z)$ is the vector force given by $\vec{F}(x,y,z)=z^2\vec{i}+y\vec{j}+xz\vec{k}$ and S is hemi-sphere $z=\sqrt{4-x^2-y^2}$.,$0\le z\le 4$.

Q:1 (a) $\sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{e^{n-1}}$ both are g.s. with c.r < 1

$$S = \frac{1/2}{1-1/2} + \frac{1}{1-1/e} = 1 + \frac{e}{e-1} \quad \#$$

1 (b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+1}}$ using limit comparison test with c'gl series $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ 1

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^5+1}}}{\frac{1}{\sqrt{n^5}}} = \lim_{n\to\infty} \frac{\sqrt{n^5}}{\sqrt{n^5+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^5}} = 1 \neq 0.$$ Both series c'ge or d'ge together 1

$\Rightarrow$ given series c'gt. 1

1 (c) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n4^n} \Rightarrow \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \frac{n4^n}{(n+1)4^{n+1}} |x|$

$= \frac{1}{4}|x| \Rightarrow$ c'gt if $\frac{1}{4}|x| < 1 \Rightarrow \boxed{-4 < x < 4}$ 1

Convergence at $x=-4$

$\sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$

d'gt $\frac{1}{2}$

Convergence at $x=4$

$\sum_{n=1}^{\infty} (-1)^n \frac{4^n}{n4^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

c'gt A.S. $\frac{1}{2}$

Interval of c'gence $-4 < x \leq 4$

Radius " " $\rho = \frac{4-(-4)}{2} = 4$ 1

1 (d) $\int_{-1}^{1}\frac{\sin x}{x}\,dx \approx \int_{-1}^{1}\frac{1}{x}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}\right]dx$ 1

$= \int_{-1}^{1}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}\right]dx = \left[x-\frac{x^3}{18}+\frac{x^5}{600}\right]_{-1}^{1}$ 1

$=\left[1-\frac{1}{18}+\frac{1}{600}\right]-\left[-1+\frac{1}{18}-\frac{1}{600}\right]$

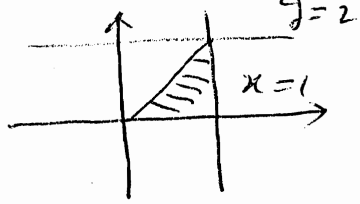
$=2-\frac{1}{9}+\frac{1}{300}=\frac{5400-300+9}{2700}=\frac{5109}{2700}$

≈ 1.8922 1

Q: 2(a) $\int_0^2\int_{y/2}^{1} e^{x^2}\,dx\,dy$ 2

$0\le y\le 2$

$y/2\le x\le 1$



$=\int_0^1\int_0^{2x} e^{x^2}\,dy\,dx=\int_0^1 e^{x^2}2x\,dx$

$0\le x\le 1$

$0\le y\le 2x$

$=\left[e^{x^2}\right]_0^1 = e-1$ 1

$z=\sqrt{1-x^2-y^2}=g(x,y)$

2(b) $S.A.=2\iint_{R_{xy}}\sqrt{1+g_x^2+g_y^2}\,dA$

$g_x=\frac{-x}{\sqrt{1-x^2-y^2}}$

$g_y=\frac{-x}{\sqrt{1-x^2-y^2}}$

$=2\iint_{R_{xy}}\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\,dA$

$x^2+y^2=y$

$r=\sin\theta$

$=2\iint_{R_{xy}}\frac{1}{\sqrt{1-x^2-y^2}}\,dA$ 1

$\boxed{0\le r\le \sin\theta \quad 0\le\theta\le\pi}$

1 $=2\int_0^{\pi}\int_0^{\sin\theta}(1-r^2)^{-1/2}r\,dr\,d\theta = -\frac{1}{2}\int_0^{\pi}\left[\frac{(1-r^2)^{1/2}}{1/2}\right]_0^{\sin\theta}d\theta$

$=2-\int_0^{\pi}[\cos\theta-1]\,d\theta = -2\left[\sin\theta-\theta\right]_0^{\pi} = 2\pi$ 1 #

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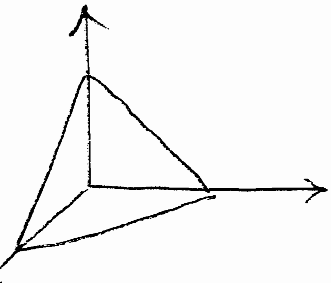


2(c) $\iiint_Q x\,dv$

$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} x\,dz\,dy\,dx$ $1\frac{1}{2}$ $= \left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$

$0 \le z \le 1-x-y$

$0 \le y \le 1-x$

$0 \le x \le 1$

$= \int_0^1\int_0^{1-x} x(1-x-y)\,dy\,dx = -\int_0^1 x \left.\frac{(1-x-y)^2}{2}\right]_0^{1-x} dx$

$= -\frac{1}{2}\int_0^1 x\left[(1-x-1+x)^2 - (1-x)^2\right]dx = +\frac{1}{2}\int_0^1 x(1-x)^2\,dx$

$= \frac{1}{2}\int_0^1 (x-2x^2+x^3)\,dx = \frac{1}{2}\left[\frac{x^2}{2}-\frac{2}{3}x^3+\frac{x^4}{4}\right]_0^1$

$= \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] = \frac{1}{2}\left[\frac{6-8+3}{12}\right] = \frac{1}{24}$ # $1\frac{1}{2}$

2(d) $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} (x^2+y^2)\,dz\,dy\,dx$ 2

$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^3\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3(4-r^2)\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 (4r^3-r^5)\,dr\,d\theta = \int_0^{2\pi}\left[r^4-\frac{r^6}{6}\right]_0^2 d\theta = \int_0^{2\pi}\left[16-\frac{32}{3}\right]d\theta$

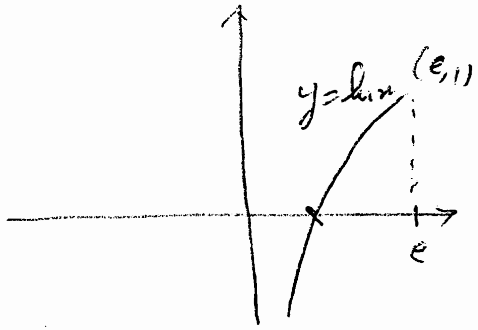
$= \frac{16}{3}(2\pi) = \frac{32\pi}{3}$ 1

Q. 3 (a) $\quad F(x,y) = \frac{y}{x}\, i + j$

$$\frac{\partial M}{\partial y} = \frac{1}{x} \neq \frac{\partial N}{\partial x} = 0 \qquad 1\tfrac{1}{2}$$

Not independent of path

C : $x = t$, $y = \ln t$, $1 \le t \le e$



$$\int_C \bar{F} \cdot d\bar{r} = \int_C \frac{y}{x}\, dx + dy$$

$$= \int_1^e \frac{\ln t}{t}\, dt + \frac{1}{t}\, dt$$

$$\cancel{= \left[-\frac{1}{t^2} \ln t + \int \frac{1}{t}\right]}$$

$$= \int_1^e \ln t \,(1/t)\, dt + \int_1^e \frac{1}{t}\, dt = (\ln t)^2\Big]_1^e + \Big[\ln t\Big]_1^e$$

$$= (1 - 0) + (1 - 0) = 2 \qquad 1\tfrac{1}{2}$$

3 (b) Greens theorem $\quad \oint_C (x-y)\, dx + x\, dy = \iint_R (1+1)\, dA \qquad 1$

$$= 2\int_0^{\pi/9}\int_0^3 r\, dr\, d\theta = 2\int_0^{\pi/9} \frac{9}{2}\, d\theta = 9(\pi/9) = \pi \qquad 1$$

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$0 \le r \le 3$

$0 \le \theta \le \pi/9$

2

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(c) $\iint_S F\cdot\bar{n}\,ds = \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dv$

L.H.S = R.H.S 1

R.H.S $\iiint_Q 3\,dv$

$0 \le \theta \le 2\pi$
$0 \le \varphi \le \pi/2$
$0 \le \rho \le 3$

$\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1\frac{1}{2}$

$= 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$

$= 3\int_0^{2\pi}\int_0^{\pi/2} \left[\frac{\rho^3}{3}\right]_0^3 \sin\varphi \, d\varphi\, d\theta = 27\int_0^{2\pi}\left[-\cos\varphi\right]_0^{\pi/2} d\theta$

$= 27\left[-0+1\right]2\pi = 54\pi$ $\frac{1}{2}$

Cylindrical
$0 \le \theta \le 2\pi$
$0 \le r \le 3$
$0 \le z \le \sqrt{9-r^2}$

$= 3\int_0^{2\pi}\int_0^3\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta = 3\int_0^{2\pi}\int_0^3 \sqrt{9-r^2}\; r\,dr\,d\theta$

$= -\frac{3}{2}\int_0^{2\pi}\left[\frac{(9-r^2)^{3/2}}{3/2}\right]_0^3 d\theta = -\frac{3}{3}\left[0-(9)^{3/2}\right]\int_0^{2\pi} d\theta$

$= 27(2\pi) = 54\pi$

L.H.S For $F = \langle x, y, z\rangle$ and $S = \{(x,y,z) \mid \cancel{x^2+y^2+z^2=9},\ z = \sqrt{9-x^2-y^2}\}$

$\iint_n F\cdot\bar{n}\,ds = \iiint_G \operatorname{div}(F)\,dv$, $z = \sqrt{9-x^2-y^2}$

L.H.S $\iint_n F\cdot\bar{n}\,ds = \iint_{R_{xy}} (-Mg_x - Ng_y + P)\,dA = \iint_R \left(\frac{x^2+y^2+9-x^2-y^2}{\sqrt{9-x^2-y^2}}\right)$ 1

$= \frac{9}{2}\int_0^{2\pi}\int_0^3 (9-r^2)^{-1/2}(-2r)\,dr\,d\theta$ 1 $= -\frac{9}{2}\int_0^{2\pi}\left[\frac{(9-r^2)^{1/2}}{1/2}\right]_0^3 d\theta$

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$$= -9\int_0^{2\pi} [0-3]\, d\theta = 27(2\pi) = 54\pi$$ 1

$$= 54\pi$$

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3 (d) $\iint_S (\text{curl} F)\cdot \bar{n}\, ds$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & y & xz \end{vmatrix} = \langle 0, z, 0 \rangle$$

$$= \iint_{R_{xy}} (-Mg_x - Ng_y + P)\, dA = \iint_{R_{xy}} \left(\frac{z\, y}{\sqrt{4-x^2-y^2}}\right) dA$$ 1

$z = g(x,y) = \sqrt{4-x^2-y^2}$

$g_y = \dfrac{-y}{\sqrt{4-x^2-y^2}}$

$$= \iint_{R_{xy}} y\, dA = \int_0^{2\pi}\int_0^2 r^2 \sin\theta\, dr\, d\theta$$ 1

$$= \int_0^{2\pi} \left.\frac{r^3}{3}\right]_0^2 \sin\theta\, d\theta = \frac{8}{3}\left[-\cos\theta\right]_0^{2\pi}$$

$$= \frac{8}{3}[-1+1] = 0$$ 1