King Saud University
Department of Mathematics
M-203[Final Examination]
(Differential and Integral Calculus)
(Second-Semester 1437/38)

Max.Marks:40 Time:3hrs

Marking Scheme: Q.No:1[2+4+3+3]; Q.No:2[4+4+4+4]; Q.No:3[3+3+3+3]

Q.No: 1 (a) Use integral test to determine whether the series $\sum_{n=2}^{\infty} \frac{\ln(n^3)}{n}$ converges or diverges.

(b) Find the interval of convergence and radius of convergence of the power Series $\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2 3^n}$.

(c) Find the power series representation of $f(x) = \ln(1+x)$, $|x|<1$ and use first three non-zero terms to calculate $\ln(1.2)$.

(d) Find the Maclaurin series of the function $f(x) = e^x$ and use its first three non-zero terms to approximate the integral $\int_0^1 e^{x^2} dx$.

Q.No: 2 (a) Evaluate the integral $\int_0^2 \int_x^2 y^4 \cos(xy^2) dy dx$.

(b) Find the area of the surface $z = 4 - x^2 - y^2$ that lies above the $xy-plane$.

(c) Use cylindrical coordinates to evaluate $\iiint_Q \sqrt{x^2+y^2} dV$, where Q is the solid bounded by the paraboloid $z = 1-(x^2+y^2)$ and the xy-plane.

(d) Use spherical coordinates to find the centroid of the solid bounded by the cone $z = \sqrt{x^2+y^2}$ and hemi-sphere $z = \sqrt{8-x^2-y^2}$.

Q.No: 3 (a) Evaluate the line integral $\int_C (y^3+3y) ds$ where $C: x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le \pi$.

(b) Show that the line integral

$$\int_{(0,1,2)}^{(1,0,3)} (2xy + z^2) dx + (x^2 - z + 2y) dy + (2zx - y) dz$$

is independent of path and find its value.

(c) Use the Green's theorem to evaluate $\oint_C (x^2 y dx + xy^2 dy)$, where C is the triangle with vertices (-1,0), (1,0), and (0,1).

(d) Use the Green's theorem to find the area of the circle $x^2 + y^2 = a^2$, $a > 0$.

Q. 1(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n^3)}{n}$ [Marks: 2]

$$f(x) = \frac{3\ln x}{x} > 0 \quad \text{on} \quad [2, \infty)$$

(i) $f(x)$ cont. on $[2, \infty)$

(ii) $f'(x) = 3\dfrac{(1 - \ln x)}{x^2} < 0$ on big $x \in [2, \infty)$ 1

$$3\int_2^{\infty} \ln x \left(\tfrac{1}{x}\right) dx = 3 \lim_{t\to\infty} \int_2^t \ln(x)\left(\tfrac{1}{x}\right) dx = \tfrac{3}{2} \lim_{t\to\infty} \Big[(\ln x)^2\Big]_2^t$$

$$= \infty \quad \text{d'gt}$$ 1

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2\, 3^n}$ [Marks: 4]

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(2x-5)^{n+1}}{(n+1)^2\, 3^{n+1}} \times \frac{n^2\, 3^n}{(2x-5)^n}\right| = \frac{n^2}{3(n+1)^2}|2x-5|$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{3}|2x-5| \Rightarrow \text{c'gt if } |2x-5| < 3$$

$$\Rightarrow \boxed{1 < x < 4}$$ 1

C'gence at $x = 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

c'gt AS 1

C'gence at $x = 4$

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

c'gt p-series 1

Interval of C'gence $1 \le x \le 4$

Radius of C'gence $\rho = \frac{3}{2}$ 1

Q:1 (c) $\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x [1 - t + t^2 - t^3 + \cdots]\,dt$

[Marks: 3]

$= t - \frac{t^2}{2} + \frac{t^3}{3} \cdots \Big]_0^x \approx x - \frac{x^2}{2} + \frac{x^3}{3}$ (1)

$\ln(1.2) \approx \ln(1 + 0.2) = (0.2) - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3}$

(1)

$= 0.2 - 0.02 + 0.00267 \approx 0.183$

(1)

Q:1 (d) $f(x) = e^x, \; f'(x) = e^x, \; f''(x) = e^x, \cdots\cdots$ [Marks: 3]

$e^x = 1 + x + \frac{x^2}{2!} + \cdots\cdots$ (1)

$e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2!} + \cdots$ (1)

$\int_0^1 e^{x^2}\,dx \approx \int_0^1 \left[1 + x^2 + \frac{x^4}{2}\right] dx = \left[x + \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1$

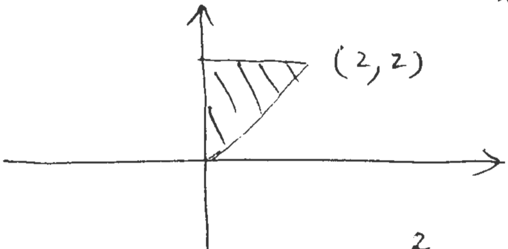
$= 1.4333$ (1)

Q, 2(a) $\int_0^2 \int_x^2 y^4 \cos(xy^2)\,dy\,dx$ $\quad 0 \le x \le 2$ [Marks: 4]

$x \le y \le 2$

$0 \le y \le 2$

$0 \le x \le y$



$= \int_0^2 \int_0^y y^4 \cos(xy^2)\,dx\,dy = \int_0^2 y^4 \frac{\sin(xy^2)}{y^2}\Big]_0^y dy$

(2)

$= \int_0^2 y^2 \sin(y^3)\,dy$ $\qquad u = y^3$

$\frac{1}{3}du = y^2\,dy$

$= \frac{1}{3}\int_0^8 \sin u\,du = -\frac{1}{3}[\cos u]_0^8 = \frac{1}{3}[1 - \cos 8]$

(2) #

[redacted]

Q: 2(b)

$$S.A. = \int_0^{2\pi}\int_0^{2}\sqrt{1+4x^2+4y^2}\,dA = \int_0^{2\pi}\int_0^{2}\sqrt{1+4r^2}\; r\,dr\,d\theta \quad [\text{Marks: } 4]$$

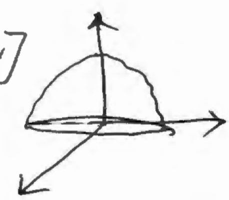
(2) (1)

$$= \frac{1}{8}\int_0^{2\pi} \frac{(1+4r^2)^{3/2}}{3/2}\Bigg]_0^2 d\theta = \frac{1}{12}\left[17^{3/2}-1\right]2\pi$$

$$= \frac{1}{6}\left(17^{3/2}-1\right)\pi \quad (1)$$

Q: 2(c)

$$\int_0^{2\pi}\int_0^{1}\int_{\sqrt{r^2}}^{1-r^2} r\,dz\,dr\,d\theta \quad (3) \quad [\text{Marks: } 4]$$



$$= \int_0^{2\pi}\int_0^{1} r^2(1-r^2)\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^3}{3}-\frac{r^5}{5}\right]_0^1 d\theta = \frac{4}{15}\pi \quad (1)$$

Q: 2(d)

$\cancel{d\varphi = 2\,dt}$ $\cancel{m = \int}$

$$m = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \quad [\text{Marks: } 4]$$

(3) (2)

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\Bigg]_0^{\sqrt{8}} \sin\varphi\, d\varphi\, d\theta = \frac{8\sqrt{8}}{3}\int_0^{2\pi} \left[-\cos\varphi\right]_0^{\pi/4} d\theta = \frac{8\sqrt{8}}{3}\left[1-\frac{1}{\sqrt{2}}\right]2\pi \quad (1)$$

To Calculate z

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^3 \sin\varphi\cos\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^4}{4}\Bigg]_0^{\sqrt{8}} \sin\varphi\cos\varphi\, d\varphi\, d\theta$$

$$= \frac{64}{4}\int_0^{2\pi} \frac{\sin^2\varphi}{2}\Bigg]_0^{\pi/4} d\theta = 8\left(\frac{1}{2}\right)2\pi = 8\pi$$

$$\bar{z} = \frac{1}{2\sqrt{8}/3\left(1-\frac{1}{\sqrt{2}}\right)}, \quad \bar{x} = 0, \quad \bar{y} = 0$$

$$= \frac{4}{3}(\sqrt{2}-1) \quad (1)$$

Q: 3(a) $ds = 2\,dt$ [Marks: 3]

$$\int_0^{\pi} (8\sin^3 t + 8\sin t)\,2\,dt = 16\int_0^{\pi} (1-\cos^2 t)\sin t\,dt + 12\int_0^{\pi} \sin t\,dt$$

2

$u = \cos t$

$$= -16\int_1^{-1} (1-u^2)\,du + 12\left[-\cos t\right]_0^{\pi}$$

$$= -16\left[u - \frac{u^3}{3}\right]_1^{-1} + 12\left[-(-1)+1\right] = -16\left[\left(-1+\frac{1}{3}\right)-\left(1-\frac{1}{3}\right)\right] + 24$$

$$= -16\left(-2+\frac{2}{3}\right) + 24 = \frac{64}{3} + 24 = \frac{136}{3}$$

1

Q: 3(b)

$M = 2xy + z^2$, $N = x^2 - z + 2y$, $P = 2zx - y$ [Marks: 3]

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = -1 = \frac{\partial P}{\partial y}$$

$\Rightarrow$ Conservative.

$$f_x = 2xy + z^2 \Rightarrow f(x,y,z) = x^2y + z^2x + c_1(y,z)$$

$$f_y = x^2 - z + 2y \Rightarrow f(x,y,z) = x^2y - zy + y^2 + c_2(x,z)$$

$$f_z = 2zx - y \Rightarrow f(x,y,z) = z^2x - yz + c_3(x,y)$$

$$\Rightarrow f(x,y,z) = x^2y + z^2x - zy + y^2 + c$$

2

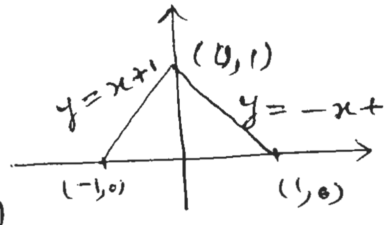
$$\int_{(0,1,2)}^{(1,0,3)} = 10$$

1

Q: 3(c)

$$\oint_C x^2 y\, dx + x y^2\, dy$$ [Marks: 3]

$$= \iint_R (y^2 - x^2)\, dA$$



$$= \int_0^1 \int_{y-1}^{1-y} (y^2 - x^2)\, dx\, dy \quad 2$$

$$= \int_0^1 \left. y^2 x - \frac{x^3}{3} \right]_{y-1}^{1-y} dy = 2 \int_0^1 \left[y^2 - y^3 + \frac{(y-1)^3}{3} \right] dy$$

$$= 2 \left[\frac{y^3}{3} - \frac{y^4}{4} + \frac{(y-1)^4}{12} \right]_0^1 = 2 \left[\left(\frac{1}{3} - \frac{1}{4} \right) - \frac{(-1)^4}{12} \right]$$

$$= 2 \left(\frac{1}{3} - \frac{1}{4} - \frac{1}{12} \right) = 2 \left(\frac{4-3-1}{12} \right) = 0$$

1

Q: 3(d)

$$A = \oint_C x\, dy \quad 1 \quad \text{[Marks: 3]}$$

$x = a\cos\theta$

$y = a\sin\theta$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} a\cos\theta \cdot a\cos\theta\, d\theta \quad 1$$

$$= a^2 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2}\, d\theta = \frac{a^2}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{2\pi}$$

$$= \frac{a^2}{2}(2\pi) = a^2\pi \quad 1$$