

King Saud University, Department of Mathematics
Math 204 (3H), 40/40, Final Exam 20/12/2018

Question 1[4,4] a) Obtain the general solution of the following Bernoulli differential equation

$$(x^3 + y^2 + 3xy^2)dx - 2xydy = 0, \quad x > 0, y \neq 0.$$

b) Solve the initial value problem

$$\begin{cases} 2e^x \cos y dx + (1 + e^x) \sin y dy = 0, & 0 < y < \frac{\pi}{2} \\ y(0) = \frac{\pi}{4}. \end{cases}$$

Question 2[4,4] a) Find the largest interval for which the following initial value problem admits a unique solution

$$\begin{cases} \sqrt[3]{x-3}y'' + (x-1)^{-1/2}y' + e^x y = (x-1)^2 \\ y(2) = -1, \quad y'(2) = 0. \end{cases}$$

b) Find the family of orthogonal trajectories of the family of curves

$$\sqrt{x} + \sqrt{y-C} = 1.$$

Question 3[4,5,5] a) Show that the functions: $f_1(x) = x$, $f_2(x) = x-1$, $f_3(x) = x+3$ are linearly dependent or linearly independent on \mathbb{R} .

b) Use power series method to find a series solution of the differential equation

$$y'' - xy = 0,$$

about the ordinary point $x = 0$, which satisfies $y(0) = 2$, $y'(0) = 3$.

c) Obtain the general solution of

$$x^2y'' - 3xy' + 3y = x^4, \quad x > 0.$$

Question 4[5,5] a) Let f be a periodic function of period 2π given by:

$$f(x) = \pi - |x| \quad \text{for } x \in [-\pi, \pi].$$

Sketch the graph of f on $(-\pi, 3\pi)$, find its Fourier series and deduce the value of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

b) Sketch the graph of the following function and find its Fourier integral

$$f(x) = \begin{cases} \cos x & \text{if } |x| \leq \pi \\ 0, & \text{if } |x| > \pi \end{cases} \text{. Deduce that } \int_0^{\infty} \frac{\lambda \sin(\lambda\pi)}{1-\lambda^2} d\lambda = \frac{\pi}{2}.$$

(Hint: $\sin(\pi - x) = \sin x$, $\sin(\pi + x) = -\sin x$).

Complete Solutions of final exam
M204, 1439/1440 H, first semester

Question ①

$$② (x^3 + y^2 + 3xy^2)dx - 2xydy = 0; \quad x > 0 \text{ and } y \neq 0$$

$$-(x^3 + y^2 + 3xy^2) + 2xyy' = 0$$

$$y' - \frac{x^3 + y^2(1+3x)}{2xy} = 0, \quad y' - \frac{1+3x}{2x}y = \frac{x^2}{2}y^{-1}$$

We have a Bernoulli's equation, with $n=-1$, then

$$y'y - \frac{1+3x}{2x}y^2 = \frac{x^2}{2}, \quad \text{we put } u = y^2 \Rightarrow \frac{u'}{2} = yy'$$

$$\frac{u'}{2} - \frac{1+3x}{2x}u = \frac{x^2}{2} \Rightarrow \boxed{u' - (\frac{1}{x} + 3)u = x^2} \text{ is a linear D.E.}$$

$$\text{D.E., hence } \mu(x) = e^{\int(\frac{1}{x}+3)dx} = \frac{e^{-3x}}{x}$$

$$\mu(x)u = u \frac{e^{-3x}}{x} = \int x^2 \cdot \frac{e^{-3x}}{x} dx = \int x e^{-3x} dx$$

$$u \frac{e^{-3x}}{x} = x(-\frac{e^{-3x}}{3}) - \frac{1}{9}e^{-3x} + C$$

$$\boxed{y^2 = -\frac{x^2}{3} - \frac{x}{9} + Cx e^{3x}}$$

This is the family of solutions

of the D.E.

$$③ \begin{cases} 2e^x \cos y dx + (1 + e^x) \sin y dy = 0 \\ y(0) = \frac{\pi}{4} \end{cases} \quad 0 < y < \frac{\pi}{2}$$

$$\frac{2e^x}{1+e^x} dx + \frac{\sin y}{\cos y} dy = 0$$

$$2 \ln(1 + e^x) - \ln(\cos y) = C$$

$$\ln \frac{(1 + e^x)^2}{\cos y} = C \quad \text{or} \quad \boxed{(1 + e^x)^2 = C_1 \cos y}, \quad C_1 = e^C$$

$$\text{From the condition } y(0) = \frac{\pi}{4} \Rightarrow 4 = C_1 \cos(\frac{\pi}{4}) = C_1 \frac{1}{\sqrt{2}}$$

$C_1 = 4\sqrt{2}$, so the solution of the IVP is

$$\boxed{(1 + e^x)^2 = 4\sqrt{2} \cos y}$$

①

Question ②

$$② \sqrt[3]{x-3} y'' + \frac{1}{\sqrt{x-1}} y' + e^x y = (x-1)^2,$$

$a_2(x) = \sqrt[3]{x-3}$ is cont. on \mathbb{R} , and $a_2(x) \neq 0$ if $x \neq 3$

$a_1(x) = \frac{1}{\sqrt{x-1}}$ is cont. on $(1, \infty)$

$a_0(x) = e^x$, and $g(x) = (x-1)^2$ are cont. on \mathbb{R}

②



But $y(2) = -1$, $y'(2) = 0$, then the largest interval for which the IVP admits a unique solution is $I = (1, 3)$

②

③ $\sqrt{x} + \sqrt{y-c} = 1$, where c is an arbitrary constant.

Solution Differentiating the equation respect to x , and we find

$$\frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y-c}} = 0, \text{ but } \sqrt{y-c} = 1 - \sqrt{x}, \text{ then}$$

$y' = -\frac{1-\sqrt{x}}{\sqrt{x}}$. Then we have to solve the D.E

①

$$y' = \frac{\sqrt{x}}{1-\sqrt{x}}, \text{ we put } t = 1 - \sqrt{x} \Rightarrow \sqrt{x} = 1 - t$$

①

$$x = (1-t)^2 \Rightarrow dx = -2(1-t)dt$$

$$y = \int -2 \frac{(1-t)^2}{t} dt = -2 \int \frac{1+t^2-2t}{t} dt$$

$$y = -2 \int \left(\frac{1}{t} + t - 2 \right) dt = -2 \ln|t| - t^2 + 4t + C$$

②

$$y = -2 \ln|1-\sqrt{x}| - (1-\sqrt{x})^2 + 4(1-\sqrt{x}) + C$$

$y = -2 \ln|1-\sqrt{x}| - x - 2\sqrt{x} + C$ This is the family of

curves which is orthogonal to the family $\sqrt{x} + \sqrt{y-c} = 1$

Question ③

$$\textcircled{a} \quad f_1(x) = x, \quad f_2(x) = x-1, \quad \text{and} \quad f_3(x) = x+3$$

We remark that $W(f_1, f_2, f_3) = \begin{vmatrix} x & x+1 & x+3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$ for all $x \in \mathbb{R}$

So these functions may be linearly independent or dependent on \mathbb{R} .

Let c_1, c_2 and $c_3 \in \mathbb{R}$ s.t. $c_1x + c_2(x-1) + c_3(x+3) = 0$ for all $x \in \mathbb{R}$.

For $x=0$, $x=1$, and $x=-1$, we have

$$\begin{cases} -c_2 + 3c_3 = 0 \\ c_1 + 4c_3 = 0 \\ -c_1 - 2c_2 + 2c_3 = 0 \end{cases} \Rightarrow \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & 4 \\ -1 & -2 & 2 \end{vmatrix} = 1(2+4) + 3(-2) = 6 - 6 = 0$$

So these equations have infinity solutions. For example, for $(c_1=1) \Rightarrow$

$$c_3 = -\frac{1}{4}, \quad c_2 = 3c_3 = \frac{-3}{4} \quad \text{and} \quad -1 + \frac{3}{2} - \frac{1}{2} = 0$$

So there exist $(1, -\frac{3}{4}, -\frac{1}{4}) \neq (0, 0, 0)$ s.t.

$(1)x - \frac{3}{4}(x-1) - \frac{1}{4}(x+3) = 0$ for all $x \in \mathbb{R}$. So f_1, f_2 and f_3 are linearly dependent on \mathbb{R}

$$\textcircled{b} \quad y'' - xy = 0, \quad y(0) = 2, \quad y'(0) = 3$$

Solution The solution of the D.E is the form $y = \sum_{n=0}^{\infty} a_n x^n$; $x \in \mathbb{R}$. From the initial conditions, we have $(a_0 = 2)$ and $(a_1 = 3)$

$$\begin{aligned} y'' - xy &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \\ n-2 &= p_k \\ n &= p_k + 2 \\ n+1 &= p_k \\ n &= p_k - 1 \end{aligned} \quad \textcircled{1}$$

$$\sum_{k=0}^{\infty} (p_k + 2)(p_k + 1)a_k x^{\frac{p_k}{2}} - \sum_{k=1}^{\infty} a_k x^{\frac{p_k}{2}} = 0$$

$$2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} - a_{k-1}] x^k = 0$$

Then $a_2 = 0$,

$$\frac{a_k}{k+2} = \frac{a_{k-1}}{(k+2)(k+1)} \quad k \geq 1$$

$$\text{For } k=1 \Rightarrow a_3 = \frac{a_2}{6} = \frac{2}{6} = \frac{1}{3}$$

(2)

$$k=2 \Rightarrow a_4 = \frac{a_3}{12} = \frac{3}{12} = \frac{1}{4}, \text{ and so on.}$$

Then the solution of the IVP is unique solution given by

$$y(x) = 2 + 3x + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots \quad ; \quad x \in \mathbb{R}$$

$$(9) \quad x^2 y'' - 3xy' + 3y = x^4, \quad x > 0$$

$$\text{Solution 1)} \quad x^2 y'' - 3xy' + 3y = 0, \quad y = x^m$$

$$[m(m-1) - 3m + 3]x^m = 0 \Rightarrow m^2 - 4m + 3 = 0$$

$$(m-3)(m-1) = 0 \Rightarrow m=1, m=3.$$

$$\text{Then } y_c = c_1 x + c_2 x^3, \text{ let } y_1 = x, \text{ and } y_2 = x^3$$

$$2) \quad y_p = y_{u_1} + y_{u_2} = x u_1 + x^3 u_2 \text{ s.t.}$$

$$\begin{cases} x u'_1 + x^3 u'_2 = 0 \\ u'_1 + 3x^2 u'_2 = x^2 \end{cases}$$

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ x^2 & 3x^2 \end{vmatrix} = -x^5, \quad (1)$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & x^2 \end{vmatrix} = x^3, \quad u_1 = \frac{W_1}{W} = \frac{-x^5}{2x^3} = -\frac{1}{2}x^2, \quad u_2 = \frac{1}{6}x^3$$

$$u_2 = \frac{W_2}{W} = \frac{x^3}{2x^3} = \frac{1}{2} \Rightarrow u_2 = \frac{1}{2}x$$

$$y_p = y_{u_1} + y_{u_2} = -\frac{1}{2}x^4 + \frac{1}{2}x^4 = \frac{1}{3}x^4 \quad (2)$$

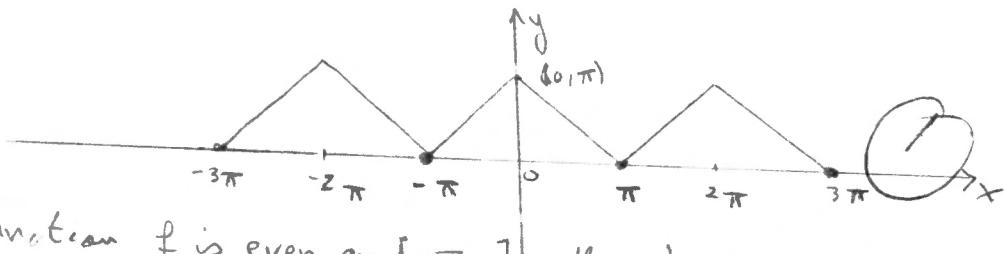
\Rightarrow the general solution of the D.E is

$$y = y_c + y_p = c_1 x + \frac{c_2}{2} x^3 + \frac{1}{3} x^4, \quad x > 0$$

Remark: We can find y_p as the form $y_p = Ax^4$, and $A = \frac{1}{3}$

Question ④

④ $f(x) = \pi - |x| ; \quad -\pi \leq x \leq \pi, \quad f(x+2\pi) = f(x), \quad x \in \mathbb{R}$



The function f is even on $[-\pi, \pi]$, then $b_n = 0, n=0, 1, 2, \dots$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \left[\pi x - \frac{x^2}{2} \right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi} \left[\frac{1}{2} \pi^2 \right] = \frac{2}{\pi}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} f(\pi - x) \cos nx dx \\ &= \frac{2}{\pi} \left[(\pi - x) \frac{\sin(nx)}{n} \right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} \frac{\sin nx}{n} (-dx) \\ &= \frac{2}{\pi} \left[-\frac{\cos nx}{n^2} \right]_0^{\pi} = \frac{-2}{\pi n^2} (\cos n\pi - 1) = \frac{2}{\pi n^2} (1 - (-1)^n) \end{aligned}$$

$$a_n = \frac{2}{\pi n^2} (1 - (-1)^n) \quad n = 1, 2, 3, \dots$$

$$\frac{f(x^+) + f(x^-)}{2} = f(x) = \pi - |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} (1 - (-1)^n) \cos nx$$

$$f(x) = \pi - |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi (2n-1)^2} \cos((2n-1)x) \quad \text{or}$$

$$f(x) = \pi - |x| = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{4}{\pi (2n+1)^2} \cos((2n+1)x)$$

$$-\pi \leq x \leq \pi$$

At $x=0$, we deduce that

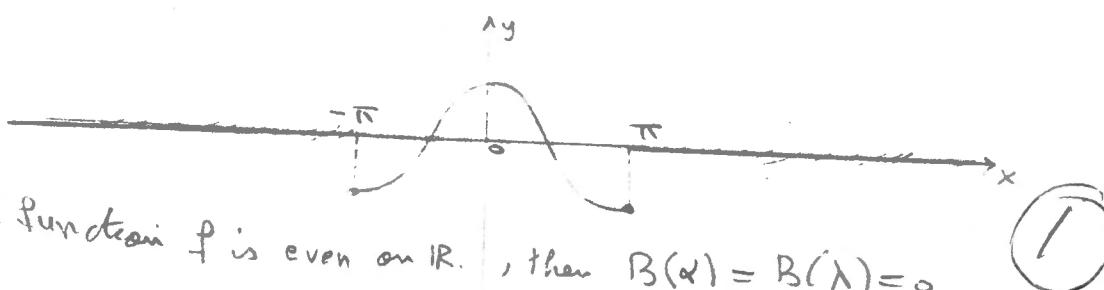
$$f(0) = \pi = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi (2n-1)^2}, \text{ then}$$

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \quad \text{or}$$

$\frac{\pi}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

(1)

(b) $f(x) = \begin{cases} \cos x & |x| \leq \pi \\ 0 & |x| > \pi \end{cases}$



The function f is even on \mathbb{R} , then $B(\alpha) = B(\lambda) = 0$.

(1)

$$A(\lambda) = \int_{-\infty}^{\infty} f(x) \cos(\lambda x) dx = 2 \int_0^{\infty} \cos x \cdot \cos \lambda x dx$$

$$= 2 \int_0^{\pi} \frac{1}{2} [\cos((1-\lambda)x) + \cos((1+\lambda)x)] dx ?$$

$$= \left[\frac{\sin((1-\lambda)x)}{1-\lambda} + \frac{\sin((1+\lambda)x)}{1+\lambda} \right]_0^{\pi};$$

$$A(\lambda) = \frac{\sin((1-\lambda)\pi)}{1-\lambda} + \frac{\sin((1+\lambda)\pi)}{1+\lambda}$$

$A(\lambda) = \frac{\sin \lambda \pi}{1-\lambda} - \frac{\sin \lambda \pi}{1+\lambda},$

(2)

$$A(\lambda) = \frac{2\lambda \sin \lambda \pi}{1-\lambda^2}, \lambda \neq 1$$

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2\lambda \sin(\lambda \pi)}{1-\lambda^2} \cos \lambda x dx$$

(3)

At $x=0$ we have

$$\frac{f(0^+) + f(0^-)}{2} = f(0) = 1 = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda\pi)}{1-\lambda^2} d\lambda$$

hence

$$\boxed{\frac{\pi}{2} = \int_0^\infty \frac{\lambda \sin(\lambda\pi)}{1-\lambda^2} d\lambda}$$

(1)