

KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS  
TIME: 3H, FULL MARKS: 40, SII /22/08/1438, MATH 204

**Question 1.** [4,5] a) Show that  $\mu(x, y) = x^{-2}y^{-3}$  is an integrating factor for the differential equation

$$(x^3y - y^2)dx - (x^4 + xy)dy = 0, \quad x > 0, y > 0.$$

b) A mass of radio active material was left in a lab. After 1 year the mass decreased by 4% and after 10 years it was found that 80 gram of the material was left. Find the initial mass and the half life of this material.

**Question 2.** a) [4,3]. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y^3 + 2x^2y}{xy^2 + x^3}, \quad y(1) = 1.$$

b) Find a linear differential equation that has the general solution

$$y = c_1 e^x + c_2 x e^x + c_3 \cos 2x + c_4 \sin 2x$$

**Question 3.** a) [4,4]. Use the undetermined coefficients method to solve the second order differential equation

$$y'' - 2y' + 2y = 4x - \cos x$$

b) Solve the differential equation

$$x^2 y'' - 2y = 2 \ln x, \quad x > 0.$$

**Question 4** [5,5]. a) Find the first six terms in a power series expansion about the ordinary point  $x_0 = 0$  for a general solution to the equation

$$y'' = xy.$$

b) Use the method of elimination to solve the system of differential equations

$$\begin{cases} x' = -3x + 2y \\ y' = -3x + 4y \end{cases}$$

**Question 5.** [6]. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the  $2\pi$ -periodic function defined by

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

Sketch the graph of  $f$  over  $(-3\pi, 3\pi)$ . Find the Fourier series of the function  $f$  and deduce the value of the series  $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ .

$$M = \frac{F \times 5}{8}$$

$$M_2 = \max\{M_1, M\}$$

## Answer Sheet

Q. a) Multiply the DE by  $\mu(x,y) = x^{-2}y^{-3}$ , we get

$$\underbrace{(xy^{-2} - x^{-2}y^{-1})}_{M} dx + \underbrace{(x^2y^{-3} + x^{-1}y^{-2})}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = -2xy^{-3} + x^{-2}y^{-2}, \quad \frac{\partial N}{\partial x} = -2xy^{-3} + x^{-2}y^{-2} \quad (1)$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{The DE is exact}) \Rightarrow \exists F(x,y)$$

$$\begin{cases} \frac{\partial F}{\partial x} = xy^{-2} - x^{-2}y^{-1} \rightarrow (1) \\ \frac{\partial F}{\partial y} = -x^2y^{-3} - x^{-1}y^{-2} \rightarrow (2) \end{cases} \quad (1)$$

$$\text{From (1), } F(x,y) = \frac{x^2}{2}y^{-2} + x^{-1}y^{-1} + \alpha(y) \rightarrow (3) \quad (1)$$

$$\text{From (3) } \frac{\partial F}{\partial y} = -x^2y^{-3} - x^{-1}y^{-2} + \alpha'(y) \rightarrow (4)$$

$$\text{From (2) and (4), we have } \alpha'(y) = 0 \Rightarrow \alpha(y) = C, \quad (1)$$

$$\text{Hence } \frac{x^2}{2}y^{-2} + x^{-1}y^{-1} = C \quad \text{is the sol of the DE.} \quad (1)$$

b) Let  $A(t)$  be the mass present at time  $t$ .

and  $A_0$  is the initial mass.

$$A(1) = A_0 - \frac{4}{100}A_0 = 0.96A_0 \rightarrow (1) \quad (1)$$

$$A(10) = 80 \rightarrow (2) \quad (1)$$

$$\text{Since } A(t) = A_0 e^{kt}$$

$$(1) \Rightarrow 0.96A_0 = A_0 e^k \Rightarrow k = \ln 0.96 \approx -0.041 \quad (1)$$

$$\text{Thus } A = A_0 e^{-0.041t}$$

$$(2) \Rightarrow 80 = A_0 e^{-0.41} \Rightarrow A_0 = 80e^{0.41} \approx 120.55. \quad (1)$$

$$\text{Half-life: } \frac{1}{2} = e^{-0.041t} \Rightarrow t = \frac{\ln 0.5}{-0.041} \approx 16.91 \text{ years.} \quad (1)$$

Q<sub>2</sub> a):  $y' = \frac{y^3 + 2xy^2}{xy^2 + x^3}$  the numerator and the denominator have the same degree, so the DE is homogeneous.

$$\frac{dy}{dx} = y' = \frac{\left(\frac{y}{x}\right)^3 + 2\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1}, \text{ let } u = \frac{y}{x} \Rightarrow y' = xu' + u. \quad (1)$$

$$\text{Hence } xu' + u = \frac{u^3 + 2u}{u^2 + 1} \Rightarrow x \frac{du}{dx} = \frac{u}{u^2 + 1} \Rightarrow \frac{(u^2 + 1) du}{u} = \frac{dx}{x}$$

$$\Rightarrow \left(u + \frac{1}{u}\right) du = \ln|x| + C \Rightarrow \frac{u^2}{2} + \ln|u| = \ln|x| + C \quad (1)$$

$$\Rightarrow \frac{u^2}{2} + \ln\left|\frac{y}{x}\right| - \ln|x| = C \Rightarrow \frac{y^2}{2x^2} + \ln\left|\frac{y}{x}\right| = C \quad (1)$$

$$\Rightarrow \exp\left\{\frac{y^2}{2x^2}\right\} \cdot \frac{|y|}{x^2} = C^* \Rightarrow \exp\left\{\frac{y^2}{2x^2}\right\} \frac{y}{x^2} = \pm C^* = C_1$$

$$\text{Since } y(1) = 1, \quad C_1 = \sqrt{e}$$

$$\text{Hence } \exp\left\{\frac{y^2}{2x^2}\right\} \frac{y}{x^2} = \sqrt{e}. \quad (1)$$

Q<sub>2</sub> b)  $m_1 = 1, m_2 = 1, m_3 = 2i, m_4 = -2i. \quad (1)$

The characteristic equation is  $(m-1)^2(m-2i)(m+2i) = 0$

After multiplication, we get:  $m^4 - 2m^3 + 5m^2 - 8m + 4 = 0 \quad (1)$

Thus the DE is  $y^{(4)} - 2y^{(3)} + 5y'' - 8y' + 4y = 0, \quad (1)$

23 a)  $y'' - 2y' + 2y = 4x - \cos x$

$$y_g = y_{gh} + y_p$$

$y_{gh} = ?$  Ch Eq:  $m^2 - 2m + 2 = 0 \Rightarrow m_1 = 1+i, m_2 = 1-i$

$$y_{gh} = e^x (C_1 \cos x + C_2 \sin x).$$

$$y_p = Ax + B + C \cos x + D \sin x$$

$$y'_p = A - C \sin x + D \cos x$$

$$y''_p = -C \cos x - D \sin x$$

By substitution in the DE.

$$-C \cos x - D \sin x - 2A + 2C \sin x - 2D \cos x + 2Ax + 2B$$

$$+ 2C \cos x + 2D \sin x = 4x - \cos x$$

$$\Leftrightarrow \cos x (C - 2D) + \sin x (D + 2C) - 2A + 2B + 2Ax = 4x - \cos x$$

Equating, we get  $A = 2, B = 2, C = -\frac{1}{5}, D = \frac{2}{5}$

$$\text{Hence } y_g = e^x (C_1 \cos x + C_2 \sin x) + 2x + 2 - \frac{1}{5} \cos x + \frac{2}{5} \sin x$$

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Q<sub>3</sub> b) :  $x^2 y'' - 2y = 2 \ln x, \quad x > 0$

$$y_g = y_{gh} + y_p$$

$x^2 y'' - 2y = 0$  : the Ch Eq<sub>1</sub>  $m^2 - m - 2 = 0 \Rightarrow m_1 = -1, m_2 = 2$

$$\Rightarrow y_{gh} = C_1 \bar{x}^{-1} + C_2 x^2$$

$$y_p = C_1(x) \bar{x}^{-1} + C_2(x) x^2, \text{ where}$$

$$\begin{cases} C_1'(x) \bar{x}^{-1} + C_2'(x) x^2 = 0 \\ -C_1'(x) \bar{x}^{-2} + 2 C_2'(x) \cdot x = \frac{2 \ln x}{x^2} \end{cases}$$

$$W = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3$$

$$C_1'(x) = \begin{vmatrix} 0 & x^2 \\ \frac{2 \ln x}{x^2} & 2x \end{vmatrix} / 3 = -\frac{2 \ln x}{3}, \quad (1)$$

$$\Rightarrow C_1(x) = -\frac{2}{3} (x \ln x - x)$$

$$C_2'(x) = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{2 \ln x}{x^2} \end{vmatrix} / 3 = \frac{2}{3} \frac{\ln x}{x^3} \quad (1)$$

$$\Rightarrow C_2(x) = \frac{2}{3} \int \frac{\ln x}{x^3} dx = -\frac{1}{3x^2} \ln x - \frac{1}{6x^2}$$

Hence  $y_g = C_1 \bar{x}^{-1} + C_2 x^2 = \frac{2}{3} (x \ln x - x) \bar{x}^{-1} - \left( \frac{1}{3x^2} \ln x + \frac{1}{6x^2} \right) x^2$

$$\boxed{y_g = C_1 \bar{x}^{-1} + C_2 x^2 + \frac{1}{2} - \ln x} \quad (1)$$

Q<sub>4</sub> a)  $y'' - xy = 0, \quad y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Hence,  $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$

$$\Leftrightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \quad (1)$$

$$\Leftrightarrow 2 a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1}] x^n = 0$$

$$\Rightarrow a_2 = 0, \quad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \quad n \geq 1 \quad (2)$$

$n=1 \quad a_3 = \frac{a_0}{1 \cdot 2}$

$$\underline{n=2}: a_4 = \frac{a_1}{12}, \underline{n=3}: a_5 = \frac{a_2}{20} = 0, \underline{n=4}: a_6 = \frac{a_0}{180}$$

$$\text{Hence } y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$= a_0 \left[ 1 + \frac{x^3}{6} + \frac{x^6}{180} + \dots \right] + a_1 \left[ x + \frac{x^4}{12} + \dots \right] \quad (2)$$

Q4 b) operator form  $\begin{cases} (D+3)[x] - 2y = 0 \rightarrow (1) \\ 3x + (D-4)[y] = 0 \rightarrow (2) \end{cases}$  (1)

To eliminate for example  $x$ , we apply  $(D+3)$  to (2) and multiply (1) by  $-3$ , we obtain:

$$(D^2 - D - 6)[y] = 0 \Rightarrow y'' - y' - 6y = 0$$

$$m^2 - m - 6 = 0 \Rightarrow m_1 = 3, m_2 = -2 \quad (1)$$

$$y(t) = C_1 e^{3t} + C_2 e^{-2t} \quad (1), \quad y' = 3C_1 e^{3t} - 2C_2 e^{-2t}$$

From (2), we have  $x(t) = \frac{-y' + 4y}{3}$

$$= -\frac{1}{3} (3C_1 e^{3t} - 2C_2 e^{-2t}) + \frac{4}{3} (C_1 e^{3t} + C_2 e^{-2t})$$

$$\Rightarrow x(t) = \frac{C_1}{3} e^{3t} + 2C_2 e^{-2t} \quad (2)$$



Q5: Since  $f$  and  $f'$  are piecewise continuous

on  $(-\pi, \pi)$  and since  $f(x+2\pi) = f(x)$ , then

$f$  has the Fourier series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^0 (x+\pi) dx = \frac{1}{2\pi} (x+\pi)^2 \Big|_{-\pi}^0 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^0 (x+\pi) \cos nx dx = \frac{1}{\pi} \left[ (x+\pi) \frac{\sin nx}{n} \Big|_{-\pi}^0 - \int_{-\pi}^0 \frac{\sin nx}{n} dx \right]$$

$$= -\frac{1}{\pi n} \left[ -\frac{\cos nx}{n} \right]_{-\pi}^0 = \frac{1}{\pi n^2} (1 - (-1)^n)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^0 (x+\pi) \sin nx dx = \frac{1}{\pi} \left[ -(x+\pi) \frac{\cos nx}{n} \Big|_{-\pi}^0 + \int_{-\pi}^0 \frac{\cos nx}{n} dx \right]$$

$$= -\frac{1}{n} - \frac{\sin nx}{n^2} \Big|_{-\pi}^0 = -\frac{1}{n}$$

$$\text{Hence } f(x) \approx \frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n^2} \cos nx - \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$= \frac{\pi}{4} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)}{(2n+1)^2} - \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

Let  $x=0$

$$f(0) \approx \frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \approx \frac{\pi^2}{8}$$

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