

King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S1/2016
Full Mark:25. Time 1H30mn
02/02/1438

Question 1[4,4]. a) Determine the local region in the xy -plane for which the following differential equation

$$\sqrt{9 - y^2} \frac{dy}{dx} = \ln(4 - x^2),$$

would have a unique solution through the origin $(0, 0)$.

b) Find the solution of the differential equation:

$$(x^2 - x - 2) \frac{dy}{dx} = (x - 2)^2 + 3y, \quad x > 2.$$

Question 2[4,4]. a) Verify that the differential equation

$$(x^2 + y^2 - 2)dx + (x^2 - 2xy)dy = 0, \quad x(x - 2y) \neq 0.$$

is not exact. Find a suitable integrating factor to convert it to an exact equation, and then solve it.

b) Solve the initial value problem

$$\begin{cases} 5xy^2y' + y^3 = 32(1 + \ln x)y^{-2}, & x > 0, y \neq 0 \\ y(1) = 1 \end{cases}$$

Question 3[4]. Solve the differential equation

$$\frac{dy}{dx} = \frac{1 - x - y}{x + y}, \quad x + y \neq 0.$$

Question 4[5]. Initially 100 mg of a radioactive substance was present. After 8 hours the mass has decreased by 4%. If the rate of decay is proportional to the amount of the substance present at time t . Find the amount of the remaining after 50 hours.

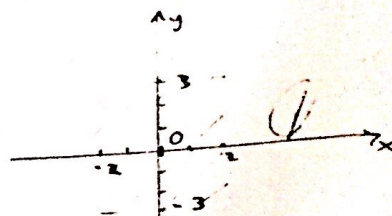
Question 1

② $y'(x) = \frac{dy}{dx} = f(x,y) = \frac{\ln(4-x^2)}{\sqrt{9-y^2}}$, $\frac{\partial f}{\partial y} = -\frac{y}{(9-y^2)^{3/2}} \cdot \ln(4-x^2)$ (2)

Thus, f and $\frac{\partial f}{\partial y}$ are continuous on $-2 < x < 2$ and $-3 < y < 3$.

As $(0,0)$ is also interior to this rectangle

$R = \{(x,y) \mid -2 < x < 2, -3 < y < 3\}$. Then (1)



The I.V.P has a unique solution defined locally, on a rectangle $R_1 \subset R$ s.t. $(0,0) \in R_1$.

③ $(x^2 - x - 2)y' = (x-2)^2 + 3y$; $x > 2$

$\frac{dy}{dx} - \frac{3}{(x+1)(x-2)}y = \frac{(x-2)}{x+1}$ which is a linear D.E

$\mu(x) = e^{\int \frac{-3}{(x+1)(x-2)} dx}$, $\frac{-3}{(x+1)(x-2)} = \frac{+1}{x+1} - \frac{1}{x-2}$ (2)

$\mu(x) = e^{-\ln(x-2) + \ln(x+1)} = e^{\ln(\frac{x+1}{x-2})} = \frac{x+1}{x-2}$ (2)

Then

$y \mu(x) = y \left(\frac{x+1}{x-2} \right) = \int \mu(x) \frac{x-2}{(x+1)} dx = \int dx$

$y \left(\frac{x+1}{x-2} \right) = x + C$

is the general solution of the D.E.

or $y = x \left(\frac{x-2}{x+1} \right) + \left(\frac{x-2}{x+1} \right) C$ (2)

Question 2

② $(x^2 + y^2 - 2)dx + (x^2 - 2xy)dy = 0$; $x(x-2y) \neq 0$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2x - 2y$, then the D.E is not exact.

$f(x) = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{4y - 2x}{x(x-2y)} = \frac{-2(x-2y)}{x(x-2y)} = \frac{-2}{x}$

$\mu(x) = e^{\int \frac{-2 dx}{x}} = e^{\ln(1/x^2)}$

$= \frac{1}{x^2}$ is an integrating factor (2)

Thus, the D.E. $(1 + \frac{y^2}{x^2} - \frac{2}{x^2})dx + (1 - \frac{2y}{x})dy = 0$ is exact

$\frac{\partial M}{\partial y} = \frac{2y}{x^2} = \frac{\partial N}{\partial x}$, hence there exists a function F of x and y s.t

$$\frac{\partial F}{\partial x} = 1 + \frac{y^2}{x^2} - \frac{2}{x^2}, \quad \frac{\partial F}{\partial y} = 1 - \frac{2y}{x}$$

$$F(x,y) = x - \frac{y^2}{x} + \frac{2}{x} + \phi(y)$$

$$\frac{\partial F}{\partial y} = \frac{-2y}{x} + \phi'(y) = 1 - \frac{2y}{x} \Rightarrow \phi'(y) = y + C$$

Then the solution of the D.E is

$$F(x,y) + \frac{-y^2}{x} + x + \frac{2}{x} + y + C = 0$$

Question 2

$$(b) \begin{cases} 5xy^2y' + y^3 = 32(1 + \ln x)y^{-2}, & x > 0, y \neq 0 \\ y(1) = 1 \end{cases}$$

Let $y + \frac{1}{5x}y = \frac{32}{5} \frac{(1 + \ln x)}{x} y^{-4}$ is Bernoulli's equation $n = -4$
 $u = y^{1-(n)} = y^5, u' = 5y^4y', \frac{u'}{5} = y^4y'$

$$(1) \quad y^4y' + \frac{1}{5x}y^5 = \frac{32}{5} \frac{(1 + \ln x)}{x}$$

$$\frac{u'}{5} + \frac{u}{5x} = \frac{32}{5} \frac{(1 + \ln x)}{x}$$

$u' + \frac{1}{x}u = 32 \left(\frac{1 + \ln x}{x} \right)$ is linear D.E. of first order

$$\mu(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$(2) \quad \mu(x)u = xu = 32 \int x \left(\frac{1 + \ln x}{x} \right) dx \\ = 32 \int (1 + \ln x) dx = 32(x + (x \ln x - x)) + C$$

$$xu = 32x \ln x + C \Rightarrow u = 32 \ln x + \frac{C}{x}$$

Then $y^5 = 32 \ln x + \frac{C}{x}$ is the solution of the D.E.

$$(3) \quad \text{Now } y(1) = 1 \Rightarrow 1 = 32 \ln(1) + C \Rightarrow C = 1$$

Thus the solution of the IVP is

$$\boxed{y^5 - 32 \ln x - \frac{1}{x} = 0}$$

Question 3

$$\frac{dy}{dx} = \frac{1-x-y}{x+y} = \frac{1-(x+y)}{x+y}, \quad x+y \neq 0$$

Let $u = x+y$, $u' - 1 = y'$, hence

$$y' = u' - 1 = \frac{1-u}{u}$$

$$u' - 1 = \frac{1-u}{u} \Rightarrow u' = 1 + \frac{1-u}{u} = \frac{1}{u}$$

$$\frac{du}{\frac{1}{u}} = dx \Rightarrow u du = dx$$

$$\frac{1}{2} u^2 = x + C \text{ or } u^2 = 2x + C_1 \text{ (2C)}$$

Thus the solution of the D.E is $\boxed{(x+y)^2 - 2x = C_1}$ or $x^2 + y^2 + 2xy - 2x = C_1$

Question 4

Let $m = m(t)$ be the mass of radio active at time t .

Then

$$\frac{dm}{dt} = -k m \Rightarrow \frac{dm}{m} = -k dt$$
$$\ln(m(t)) = -kt + C$$
$$m(t) = e^{-kt} \cdot e^C = e^{-kt} C_1; \quad (e^C = C_1)$$

We have $m(0) = 100 \text{ m.g.} \Rightarrow m(t) = 100 e^{-kt}$. But $m(8) = 100 - 4 = 96$,

$$\text{then } m(8) = 96 = 100 e^{-8k} \Rightarrow 0.96 = e^{-8k}, \text{ or } k = \frac{1}{8} \ln(0.96)$$

$$k \approx -5.1027$$
$$\boxed{m(t) \approx 100 e^{-(0.0051)t}}$$

So $m(50) \approx 100 e^{(-0.0051)(50)} = 100 e^{-0.255} \approx \boxed{77.4916 \text{ m.g.}}$