

**Question 1[4].** Find and sketch the largest local region of the  $xy$ -plane for which the initial value problem

$$\begin{cases} \sqrt{4 - y^2} \frac{dy}{dx} = \sqrt{x^2 - 1} \\ y(-2) = 0, \end{cases}$$

has a unique solution.

**Question 2[4+4]. a)** Solve the initial value problem

$$\begin{cases} y' = \frac{x}{ye^y + e^y}, \\ y(0) = 1. \end{cases}$$

**b)** Obtain the general solution of the differential equation

$$(y - 2\sqrt{xy})dx + xdy = 0, \quad x > 0, y > 0.$$

**Question 3[4+4]. a)** Solve the differential equation

$$y \left( \frac{1}{1+x^2} - ye^x \right) dx = (\tan^{-1} x) dy, \quad x > 0.$$

**b)** Find the general solution of the differential equation

$$(2x + y + 1)dx + (2y + x + 1)dy = 0.$$

**Question 4[5].** A radioactive substance that is used in medical radiology, has a half-life of 5.3 years. Suppose that an initial sample of this substance has a mass of 100 grams. Find the proportionality decay constant and find an expression for the amount of the substance at any time. How long will it take for 90% of the substance to decay?.

Answer Sheet  
MID 1 M204

(P1)

Q1.  $\begin{cases} \sqrt{4-y^2} & y' = \sqrt{x^2-1} \\ (*) & y(-2)=0 \end{cases}$

$$f(x,y) = \frac{\sqrt{x^2-1}}{\sqrt{4-y^2}}$$

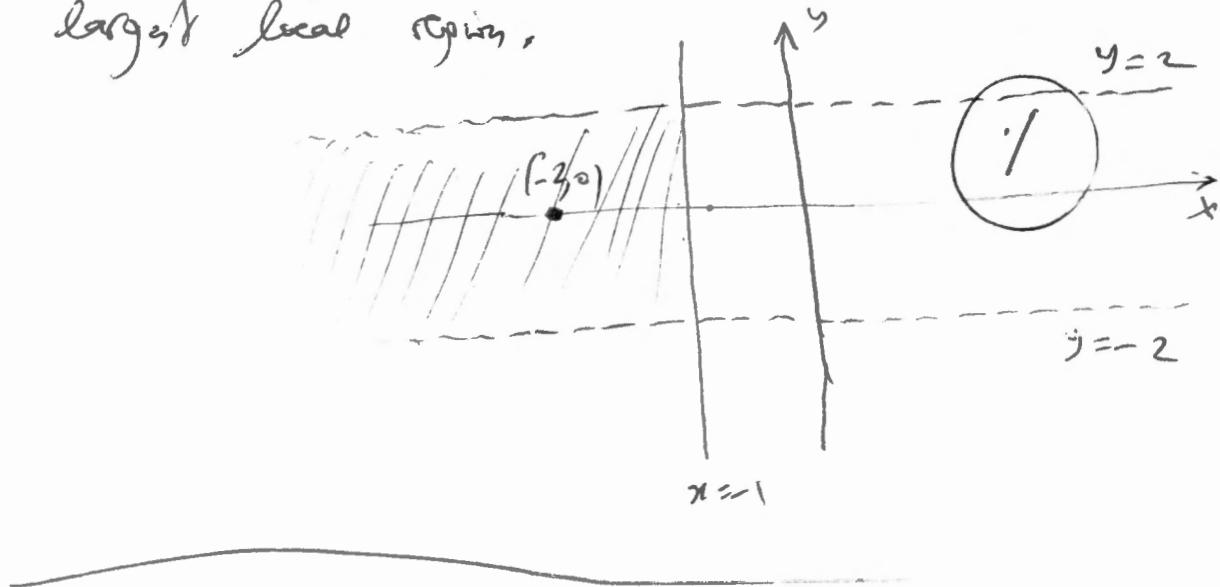
$f$  is continuous on  $D_1 = \{(x,y) \in \mathbb{R}^2 : |x| \geq 1, |y| \leq 2\}$  (1)

$\frac{\partial f}{\partial y} = \frac{y\sqrt{x^2-1}}{(4-y^2)^{3/2}}$  is continuous also on  $D_1$ .

$$D_1 = \{(x,y) \in \mathbb{R}^2 : x \in (-\infty, -1], -2 \leq y \leq 2\} \cup \{(x,y) \in \mathbb{R}^2 : x \in [1, \infty), -2 \leq y \leq 2\}$$

But  $(-2, 0) \in D^* = \{(x,y) \in \mathbb{R}^2 : x \in (-\infty, -1], -2 \leq y \leq 2\}$ . (1)

Thus (\*) has a unique solution in  $D^*$  which is the largest local region.



$$\underline{Q_2} \quad a) \quad \begin{cases} y' = \frac{x}{y e^{y+x^2}} = \frac{x}{y e^y \cdot e^{x^2}} \\ y(0)=1 \end{cases}$$

(P2)

$$\text{The DE is separable: } y e^y dy = x e^{-x^2} dx \quad (1)$$

By integration by parts:

$$\int y e^y dy = y e^y - e^y$$

$$\int x e^{-x^2} dx = -\frac{1}{2} \int d(e^{-x^2}) = -\frac{1}{2} e^{-x^2} \quad (1)$$

$$\text{Hence the solution: } (y-1)e^y + \frac{1}{2} e^{-x^2} = C$$

$$y(0)=1 \Rightarrow C = \frac{1}{2}$$

$$\text{Thus the sol of the I.V.P is } (y-1)e^y + \frac{1}{2} e^{-x^2} = \frac{1}{2} \quad (2)$$

$$b) \underbrace{(y-2\sqrt{xy})}_{M(x,y)} dx + \underbrace{x dy}_{N(x,y)} = 0, \quad x > 0, y > 0$$

$M(x,y), N(x,y)$  have the same degree, so the DE is homogeneous

The substitution  $y=ux$  transforms the DE to: (1)

$$\frac{dx}{x} + \frac{du}{2(u-\sqrt{u})} = 0 \quad (2)$$

$$\text{Integration gives: } \ln x + \ln(\sqrt{u}-1) = \ln C \quad (1)$$

$$\text{Hence the general solution } \sqrt{xy} = x + C$$

$$\text{Q3 a) } y \left( \frac{1}{1+x^2} + ye^x \right) dx = (\tan^{-1} x) dy, \quad x > 0$$

(P3)

The DE can be written as,

$$y' - \frac{y}{(1+x^2)\tan^{-1} x} = -\frac{e^x}{\tan^{-1} x} y^2 \quad (\text{Bernoulli eq}) \quad (1)$$

$$\Rightarrow y'y^2 - \frac{y^1}{(1+x^2)\tan^{-1} x} = \frac{-e^x}{\tan^{-1} x}$$

Let  $w = y^1 \Rightarrow w' = -y^2 y'$ , then we get the DE

$$w' + \frac{w}{(1+x^2)\tan^{-1} x} = \frac{-e^x}{\tan^{-1} x} \quad (\text{Linear Eq}) \quad (1)$$

$$\mu(x) = e^{\int \frac{dx}{(1+x^2)\tan^{-1} x}} = e^{\ln \tan^{-1} x} = \tan^{-1} x \quad (\text{since } x > 0)$$

$$\text{Then we have } \frac{d}{dx}(\omega \tan^{-1} x) = e^x$$

$$\Rightarrow \omega \tan^{-1} x = e^x + C$$

$$\text{Hence the general solution } \frac{\tan^{-1} x}{y} = e^x + C$$

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$$\text{b) } \underbrace{(2x+y+1)}_M dx + \underbrace{(2y+x+1)}_N dy = 0 \rightarrow (1)$$

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x} \Rightarrow \text{the DE is exact} \Rightarrow \exists F(x, y). \quad (1)$$

$$\left\{ \begin{array}{l} \frac{\partial F}{\partial x} = 2x + y + 1 \rightarrow (2) \\ \frac{\partial F}{\partial y} = 2y + x + 1 \rightarrow (3) \end{array} \right.$$

$$\text{From (2)} \quad F(x, y) = x^2 + yx + x + C(y) \rightarrow (4)$$

$$\Rightarrow \frac{\partial F}{\partial y} = x + C'(y) = 2y + x + 1 \Rightarrow C'(y) = y + 1$$

$$\text{Thus } F(x, y) = x^2 + yx + x + y + 1 = C$$

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Q5: Mathematical model  $\left\{ \begin{array}{l} \frac{dA}{dt} = KA \\ A(0) = 100 \text{ grams} \end{array} \right.$

P4

where  $A(t)$  is the quantity of the radioactive material

$$\frac{dA}{A} = Kdt \Rightarrow A(t) = C e^{Kt} = 100 e^{Kt}$$

$$50 = 100 e^{5.3K} \Rightarrow 5.3K = -\ln 2 \Rightarrow K \approx \frac{-\ln 2}{5.3} \approx -0.131$$

The amount of the substance at any time  $t$ :

$$A(t) = 100 e^{-\frac{\ln 2 t}{5.3}} = 100 e^{-0.131t}$$

If 90% of the substance decays, then 10%,

That is 10 grams remains, then

$$100 e^{-0.131t} = 10$$

$$\text{which gives } t = \frac{\ln(0.1)}{-0.131} \approx 17.6 \text{ years}$$

So it will take 17.6 years for 90% to decay of the material.

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