Differential and Integral Calculus (MATH-205)

Final Exam/Spring 2022	Time Allowed: 180 Minutes
Date: Tuesday, June 7, 2022	**Maximum Marks:** 40

Note: Attempt all **9** questions and give **DETAILED** solutions. Make sure your solutions are clearly written and contain all necessary details.

Question 1: (4°) Determine whether the following infinite series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{\ln n}}$$

Question 2: (6°) Given $f(x) = \frac{x}{(x^2+1)^2}$. Find power series representation of $f(x)$. Hence, find the radius and interval of convergence of this power series.

Question 3: (3°) Let $\mathbf{a} =< 4, 2, -1 >$ and $\mathbf{b} =< 1, 2, -3 >$. Show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Hence, find the area of the parallelogram whose adjacent sides are $\mathbf{a}$ and $\mathbf{b}$.

Question 4: (3°) Find the distance from $A(2, -6, 1)$ to the line passing through $B(3, 4, -2)$ and $C(7, -1, 5)$.

Question 5: (4°) Consider the space curve C defined by

$$C: \;\; x = 4\sqrt{1-t},\; y = t^2 - 1,\; z = \frac{4}{t},\; t < 1$$

Find parametric equations for the tangent line to C at the point $t = -1$.

Question 6: (5°) Let $z = f(x, y)$ be defined implicitly as a function of x and y by the equation

$$x\, e^y + y\, e^z = 2 + 3\ln 2 - 2\ln x.$$

Find the directional derivative of f at $(1, \ln 2)$ in the direction of maximum increase in f.

—- PTO —-

Question 7: (5°) Find the extrema and saddle points of $f(x,y) = x^2 - 3xy - y^2 + 2y - 6x$ on $R = \{(x,y) : |x| \leq 3,\ |y| \leq 2\}$. Sketch R.

Question 8: (5°) Evaluate the double integral $\int_0^2 \int_x^2 y^4 \cos(xy^2)\, dy\, dx$. Sketch and describe the region R in this integral.

Question 9: (5°) Sketch the region R and evaluate the following double integral

$$\iint_R (x+y)\, dA,$$

where R is the region bounded by $x^2 + y^2 = 2y$.

—- Good Luck —-

Pp1

Sol/F/SP22

Q.1 (4) Here $a_n = \dfrac{1}{n\sqrt[3]{\ln n}} = f(n)\,, \quad n \geq 2$ (1)

$$\int_2^\infty \frac{dx}{x\sqrt[3]{\ln x}} = \lim_{T\to\infty} \int_2^T \frac{dx}{x.\sqrt[3]{\ln x}} \quad \text{——} \; (I)$$

$$\int \frac{dx}{x.\sqrt[3]{\ln x}} = \int (\ln x)^{-1/3} . \frac{dx}{x} = \frac{(\ln x)^{2/3}}{2/3}$$

(2)

$$\therefore (I) \Rightarrow \int_2^\infty \frac{dx}{x\sqrt[3]{\ln x}} = \lim_{T\to\infty} \left| \frac{3}{2} . (\ln x)^{2/3} \right|_2^T = \frac{3}{2}\left[\infty - (\ln 2)^{2/3}\right] = \infty$$

$\therefore \displaystyle\int_2^\infty \frac{dx}{x\sqrt[3]{\ln x}}$ diverges. Hence by integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{\ln n}}$ also diverges. (1)

Q.2 (6)

Given function: $f(x) = \dfrac{x}{(1+x^2)^2}$ —(1)

$\because \dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots, \quad |x| < 1$ (1/2)

Replacing x by $-x^2$,

$\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n . x^{2n} + \cdots, \quad |x^2| < 1$ (1/2)

Diff. both sides, w.r.t. x,

$\dfrac{-2x}{(1+x^2)^2} = -2x + 4x^3 - 6x^5 + \cdots + 2n(-1)^n . x^{2n-1} + \cdots$

$\Rightarrow \dfrac{x}{(1+x^2)^2} = x - 2x^3 + 3x^5 - \cdots + n(-1)^{n+1} . x^{2n-1} + \cdots$

$= \sum_{n=1}^{\infty} (-1)^{n+1} . n . x^{2n-1}$ —(2) (2)

This is the required power series representation of (I).

For Radius & Interval of Convergence:

$u_n = (-1)^{n+1} . n . x^{2n-1}$

$\therefore$ Ratio: $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{(n+1) . x^{2n+1}}{n . x^{2n-1}}\right| = \left(1 + \dfrac{1}{n}\right) . |x^2|$

$\lim_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = |x^2|$, by ratio test for absolute convergence,

(2) converges if $|x^2| < 1$, i.e., $-1 < x < 1$. $\Rightarrow RC = 1$ (1)

At $x = 1$

$\sum_{n=1}^{\infty} (-1)^{n+1} . n . x^{2n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} . n$, which is divergent A.S.

At $x = -1$

$\sum_{n=1}^{\infty} (-1)^{n+1} . n . x^{2n-1} = \sum_{n=1}^{\infty} (-1)^{3n} . n$, which is also a divergent A.S.

$\therefore IC = (-1, 1)$ (2)

Q.3 $\vec{a} = [4,2,-1]$, $\vec{b} = [1,2,-3]$

(3)

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & -1 \\ 1 & 2 & -3 \end{vmatrix} = -4\hat{i} + 11\hat{j} + 6\hat{k} \quad (1)$$

$\vec{a}\cdot(\vec{a}\times\vec{b}) = -16+22-6 = 0 \quad \therefore \vec{a} \perp \vec{a}\times\vec{b}$ (1)

$\vec{b}\cdot(\vec{a}\times\vec{b}) = -4+22-18 = 0 \quad \therefore \vec{b} \perp \vec{a}\times\vec{b}$

Area of llgram $= \|\vec{a}\times\vec{b}\| = \sqrt{173}$ sq. units. (1)

Q.4 A(2, −6, 1), B(3, 4, −2), C(7, −1, 5)

(3)

Director of line through B & $\vec{C}$ $= \langle 4, -5, 7\rangle = \vec{BC}$ (1)

Also $\vec{BA} = \langle -1, -10, 3\rangle$, and $\vec{BC}\times\vec{BA} = 55\hat{i} - 19\hat{j} - 45\hat{k}$

$$\therefore d = \frac{\|\vec{BC}\times\vec{BA}\|}{\|\vec{BC}\|} = \frac{\sqrt{5411}}{\sqrt{90}} = \sqrt{\frac{5411}{90}} \text{ units.} \quad (2)$$

Q.5 $C: \; x = 4\sqrt{1-t}, \; y = t^2-1, \; z = \frac{4}{t}, \; t<1$

(4)

It is determined by the v' function

$\vec{r}(t) = 4\sqrt{1-t}\,\hat{i} + (t^2-1)\,\hat{j} + \frac{4}{t}\hat{k}, \; t<1$ (1)

At $t=-1$, $\vec{r}(-1) = 4\sqrt{2}\,\hat{i} + 0.\hat{j} - 4\hat{k}$, It is pt. of tangency $(4\sqrt{2}, 0, -4)$. Director of tangent line to C at pt. 't', (1)

$\dot{\vec{r}}(t) = \frac{-2}{\sqrt{1-t}}\hat{i} + 2t\,\hat{j} - \frac{4}{t^2}.\hat{k}$ (1)

At $t=-1$; $\dot{\vec{r}}(-1) = -\sqrt{2}\,\hat{i} - 2\hat{j} - 4\hat{k}$, It is D'vector of tangent line to C at pt. $t=-1$. Therefore, parametric eqns of tangent line are

$l: \; x = 4\sqrt{2} - \sqrt{2}\,s, \; y = -2s, \; z = -4-4s, \; s\in\mathbb{R}.$ (1)

Q.6 (5)   P#3

Sol/SP22/F

$$x\cdot e^{y} + y\cdot e^{z} = 2 + 3\ln 2 - 2\ln x \quad \text{—(1)}$$

$$\Rightarrow F(x,y,z) = xe^{y} + y\cdot e^{z} + 2\ln x - 2 - 3\ln 2 = 0$$

defines $z = f(x,y)$ implicitly.

$$\text{grad } f = \nabla f = f_x \hat{i} + f_y \hat{j}$$

$$f_x = -\frac{F_x}{F_z} = -\frac{\left(e^{y} + \frac{2}{x}\right)}{ye^{z}} \; ; \; f_y = -\frac{F_y}{F_z} = -\frac{(xe^{y} + e^{z})}{y\cdot e^{z}}$$

$$\therefore \text{grad } f = \nabla f = -\frac{\left(e^{y} + \frac{2}{x}\right)}{y\cdot e^{z}}\cdot\hat{i} - \frac{(xe^{y} + e^{z})}{y\cdot e^{z}}\hat{j} \quad (2)$$

At $P(1, \ln 2)$,

$$\text{grad } f\Big|_{(1,\ln 2)} = -\frac{4}{3\cdot\ln 2}\hat{i} - \frac{5}{3\ln 2}\hat{j} \quad (1)$$

$\because$ At $x = 1, y = \ln 2$, (1) $\Rightarrow z = \ln 3$

A unit vector in the direction of $\text{grad } f\Big|_{(1,\ln 2)}$ is

$$\hat{u} = [4\hat{i} + 5\hat{j}] \times \frac{-1}{\sqrt{41}}$$

$\because f$ increases rapidly in the direction of ∇f.

$$\therefore D_u f\Big|_{(1,\ln 2)} = \nabla f\Big|_{(1,\ln 2)}\cdot\hat{u} = \frac{16}{3\sqrt{41}\cdot\ln 2} + \frac{25}{3\ln 2\cdot\sqrt{41}} = \frac{\sqrt{41}}{3\cdot\ln 2} \quad (2)$$

Q.7 (5)

P#4

S.1/F/SP22

$$f(x,y) = x^2 - 3xy - y^2 + 2y - 6x$$

$$R = \{(x,y) : |x| \le 3,\ |y| \le 2\}$$

$f_x = 2x - 3y - 6$, $f_y = -3x - 2y + 2$

For critical pt. $f_x = 0$, $f_y = 0$

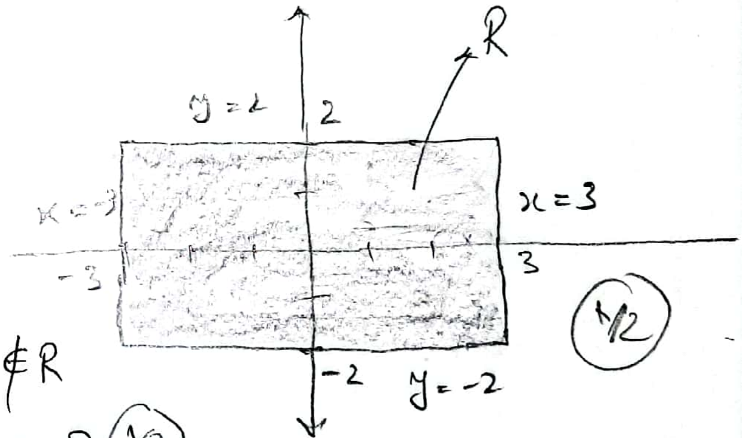
i.e., $\left.\begin{array}{l} 2x - 3y = 6 \\ 3x + 2y = 2 \end{array}\right\} \Rightarrow (x,y) = \left(\frac{18}{13}, \frac{-14}{13}\right) \in R$ is the only critical pt. inside R. (1)

$f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = -3$

$\therefore D(x,y) = \begin{vmatrix} 2 & -3 \\ -3 & -2 \end{vmatrix} = -4 - 9 = -13 < 0$

$\therefore$ $f(x,y)$ has a <u>saddle point</u> at $\left(\frac{18}{13}, -\frac{14}{13}\right)$. (1)

<u>For B'dary Extrema:</u>



(1/2)

Along $x = 3$;

$f(3,y) = 9 - 9y - y^2 + 2y - 18$

$= -y^2 - 7y - 9 = g_1(y)$

$g_1'(y) = 0 \Rightarrow -2y - 7 = 0 \Rightarrow y = -7/2 \notin R$

$\therefore$ $f(x,y)$ has no extrema on $x = 3$ in R. (1/2)

Along $x = -3$;

$f(-3,y) = 9 + 9y - y^2 + 2y + 18 = -y^2 + 11y + 27 = g_2(y)$

$g_2'(y) = -2y + 11 = 0 \Rightarrow y = +11/2 \notin R$

$\therefore$ $f(x,y)$ has no extrema on $x = -3$ in R. (1/2)

min value $= f(0,-2) = -8$ (1)

Along $y = 2$;

$f(x,2) = x^2 - 6x - 4 + 4 - 6x = x^2 - 12x = h_1(x)$ (1/2)

$h_1'(x) = 0 \Rightarrow 2x - 12 = 0 \Rightarrow x = 6 \notin R$ $\therefore$ no extrema along $y = 2$.

Along $y = -2$

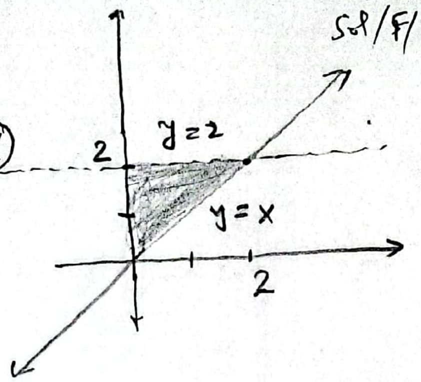
$f(x,-2) = x^2 + 6x - 4 - 4 - 6x = x^2 - 8 = h_2(x)$

$h_2'(x) = 0 \Rightarrow x = 0 \in R$

$h_2''(x) = 2 > 0$, $\therefore$ $h_2(x)$ has a min. at $(0,-2)$. ↑

Q.8. (5) $\int_0^2 \int_x^2 y^4 . \cos(xy^2) . dy . dx$

R is a region bdd b/w the lines $y = x$, $y = 2$ & $x = 0$. (2)



Reversing the order of integration, we get

$$\int_0^2 \int_x^2 y^4 . \cos(xy^2) . dy\, dx$$

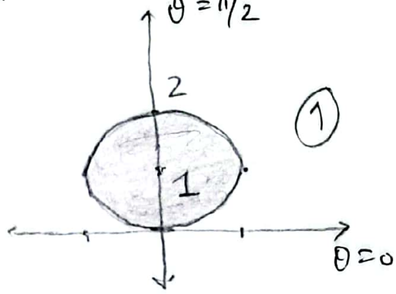
$$= \int_0^2 \int_0^y y^4 . \cos(xy^2) . dx\, dy = \int_0^2 y^4 . \left| \frac{\sin xy^2}{y^2} \right|_0^y . dy \quad (3)$$

$$= \int_0^2 y^2 \sin y^3 . dy = \left| -\frac{\cos y^3}{3} \right|_0^2 = \frac{-\cos 8 + 1}{3} \quad \text{Ans.}$$

Q.9 (5) $\iint_R (x+y) . dA = ?$

$R: \; x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$

i.e, R is a unit circle with center at $(0,1)$.



(1)

In polar form

$f(x,y) = x + y$

$\Rightarrow f(r,\theta) = r(\cos\theta + \sin\theta)$ (1)

$R_\theta: \; x^2 + y^2 = 2y \Rightarrow r = 2\sin\theta$

π/2 2 (1)

Q.9 (Ctnd)

$$\therefore \iint_R (x+y) . dA = 2 \int_0^{\pi/2} \int_0^{2\sin\theta} r(\cos\theta + \sin\theta) . r . dr . d\theta$$

$$= 2 . \int_0^{\pi/2} (\cos\theta + \sin\theta) . \left| \frac{r^3}{3} \right|_0^{2\sin\theta} . d\theta = \frac{2}{3} \int_0^{\pi/2} 8(\sin^3\theta . \cos\theta + \sin^4\theta) . d\theta$$

$$= \frac{16}{3} \left\{ \int_0^{\pi/2} \sin^4\theta . d\theta + \left| \frac{\sin^4\theta}{4} \right|_0^{\pi/2} \right\} = \frac{16}{3} \int_0^{\pi/2} \left(\frac{1 - \cos 2\theta}{2} \right)^2 . d\theta$$

$$= \frac{4}{3} \int_0^{\pi/2} (1 + \cos^2 2\theta - 2\cos 2\theta) . d\theta = \frac{4}{3} \int_0^{\pi/2} \left(1 + \frac{1 + \cos 4\theta}{2} - 2\cos 2\theta\right) d\theta$$

$$= \frac{2}{3} \int_0^{\pi/2} (3 + \cos 4\theta - 4\cos 2\theta) . d\theta = \frac{2}{3} \left| 3\theta + \frac{\sin 4\theta}{4} - 2\sin 2\theta \right|_0^{\pi/2}$$

$$= \frac{2}{3} \times \frac{3\pi}{2} = \pi . \quad \text{Ans}$$