Differential and Integral Calculus (MATH-205)

MT Exam/Spring 2022 — Time Allowed: 120 Minutes

Date: Monday, April 4, 2022 — **Maximum Marks:** 30

Note: Attempt all SIX questions and give detailed solutions. Read statements of the questions carefully and make sure you have answered each question completely.

Question 1:(4°) Write down the first five terms of the following sequence

$$\left\{(-1)^n \frac{\sqrt{n}\,\sin\sqrt{n}}{n+1}\right\}_{n=1}^{\infty}.$$

Find the limiting value of the sequence as $n \to \infty$. Is this sequence divergent?

Question 2:(6°) Find the interval of convergence of the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

Identify the function this power series represents.

Question 3:(5°) Approximate $\int_0^{\frac{1}{2}} x \cos x^3 \, dx$ using first three non-zero terms of the Maclaurin series. Use 5 decimal point accuracy in your working.

Question 4:(5°) Given $\mathbf{a} =< 2, 0, -1 >$, $\mathbf{b} =< -3, 1, 0 >$, and $\mathbf{c} =< 1, -2, 4 >$. Find the angle (in degree) between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ and the component of $\mathbf{a} \times \mathbf{b}$ along $\mathbf{c}$.

Question 5:(6°) Show that the following two planes are not parallel.

$$p_1: \; x - 5y + 3z = 11 \qquad \text{and} \qquad \mathrm{p}_2: \; -3\mathrm{x} + 2\mathrm{y} - 2\mathrm{z} = -7.$$

Find parametric equations of the line of intersection of these planes.

Question 6:(4°) Find $\mathbf{r}(t)$ subject to the given conditions

$$\ddot{\mathbf{r}}(t) = \frac{t}{(1+t^2)^2}\,\hat{\mathbf{i}} + \frac{1}{(1+t)^2}\,\hat{\mathbf{j}} + 2^{3t}\,\hat{\mathbf{k}}, \;\; \mathbf{r}(0) = 6\,\hat{\mathbf{i}} + \hat{\mathbf{j}}, \;\; \dot{\mathbf{r}}(0) = \frac{1}{2}\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \frac{2}{3\ln 2}\,\hat{\mathbf{k}}$$

—- Good Luck —-

Q.1 (4) Given Sequence $\left\{ \frac{(-1)^n.\sqrt{n}.\sin\sqrt{n}}{n+1} \right\}_{n=1}^{\infty}$

nth term: $a_n = \frac{(-1)^n.\sqrt{n}.\sin\sqrt{n}}{n+1}$, $n \geq 1$

1st Term: $a_1 = -\sin 1$; 2nd Term: $a_2 = \frac{\sqrt{2}.\sin\sqrt{2}}{3}$

3rd Term: $a_3 = -\frac{\sqrt{3}.\sin\sqrt{3}}{4}$; 4th Term: $a_4 = \frac{2\sin 2}{5}$

5th Term: $a_5 = \frac{-\sqrt{5}.\sin\sqrt{5}}{6}$ (1)

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^n.\sqrt{n}.\sin\sqrt{n}}{n+1} \right| = \lim_{n\to\infty} \frac{\sqrt{n}.\sin\sqrt{n}}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} . \lim_{n\to\infty} \frac{\sin\sqrt{n}}{\sqrt{n}} \quad \text{---} (1)$$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$\because 0 \leq \sin\sqrt{n} \leq 1 \Rightarrow 0 \leq \frac{\sin\sqrt{n}}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$

$\Rightarrow 0 \leq \lim_{n\to\infty} \frac{\sin\sqrt{n}}{\sqrt{n}} \leq \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\Rightarrow$ By Sandwich theorem, $\lim_{n\to\infty} \frac{\sin\sqrt{n}}{\sqrt{n}} = 0$

$\therefore (1) \Rightarrow \lim_{n\to\infty} |a_n| = 1 \times 0 = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0 \qquad \because \lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

$\therefore$ The given Sequence Converges to 0, Its limiting value is 0. (3)

Q.2 Given Power Series $\sum_{n=0}^{\infty} \frac{(-1)^n . x^{n+1}}{n+1}$ —— (1)

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n^{th} term: $u_n = \frac{(-1)^n . x^{n+1}}{n+1}$,

Ratio: $\frac{u_{n+1}}{u_n} = -\frac{n+1}{n+2} . x \Rightarrow \left|\frac{u_{n+1}}{u_n}\right| = \frac{n+1}{n+2} . |x|$

$\Rightarrow \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |x|$

By ratio test for AC, the given power series is AC if $|x| < 1$ & it is divergent if $|x| > 1$.

For $|x| = 1$, i.e., for $x = \pm 1$, we check separately.

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For $x = 1$:

(1) $\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ It is an alternating harmonic series.

The corresponding absolute term series is $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{n+1}\right| = \sum_{n=0}^{\infty} \frac{1}{n+1}$

It is a divergent harmonic series, now, use the AST.

C-I. $n+1 < n+2$, $\forall n \geq 0$

$\Rightarrow \frac{1}{n+1} > \frac{1}{n+2}$, $\forall n \geq 0 \Rightarrow a_n > a_{n+1}$, $\forall n \geq 2$

$\therefore$ C-I is satisfied.

C-II $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$ & C-II is satisfied.

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$\therefore$ Series (1) converges conditionally (CC) for $x = 1$.

For $x = -1$:

(1) $\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{-1}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$

$\because \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges $\therefore -\sum_{n=0}^{\infty} \frac{1}{n+1}$ also diverges.

$\therefore$ IC $= (-1, 1]$ 1

Given power series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n . x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

we know that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad , \quad |x| < 1$$

$$\Rightarrow \int \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$\because$ I.C. = $(-1,1]$.

$$\Rightarrow \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n . x^{n+1}}{n+1} , \quad -1 < x \leq 1$$

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$\therefore$ Given power series represents $\ln(1+x)$.

Q3 $\int_0^{1/2} x . \cos x^3 . dx \approx ?$

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we know that Maclaurin's series of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\Rightarrow \cos x^3 = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots$$

$$\Rightarrow x . \cos x^3 = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \cdots$$

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$$\Rightarrow x . \cos x^3 \approx x - \frac{x^7}{2!} + \frac{x^{13}}{4!}$$

(Taking only 1st 3 non-zero terms).

$$\Rightarrow \int_0^{1/2} x . \cos x^3 . dx \approx \int_0^1 \left(x - \frac{x^7}{2!} + \frac{x^{13}}{4!}\right) . dx = \left[\frac{x^2}{2} - \frac{x^8}{2! 8} + \frac{x^{14}}{4! 14}\right]_0^1$$

$$\Rightarrow \int_0^1 x . \cos x^3 . dx \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{336} = 0.5 - 0.06250 + 0.00298$$

$$\Rightarrow \int_0^1 x . \cos x^3 . dx \approx 0.44048$$

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Q.4 $\vec{a} = [2,0,-1]$, $\vec{b} = [-3,1,0]$, $\vec{c} = [1,-2,4]$

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$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{vmatrix} = \hat{i} - 3\hat{j} + 2\hat{k}$$

let θ be the angle b/w the vectors $\vec{a}\times\vec{b}$ & $\vec{c}$.

$$\therefore \cos\theta = \frac{(\vec{a}\times\vec{b})\cdot\vec{c}}{\|\vec{a}\times\vec{b}\|\,\|\vec{c}\|} = \frac{[1,-3,2]\cdot[1,-2,4]}{\sqrt{14}\cdot\sqrt{21}} = \frac{15}{\sqrt{294}} = 0.875$$

$$\Rightarrow \quad \theta = \cos^{-1}(0.875) = 28.96^\circ \approx 29^\circ \quad 3$$

$$\mathrm{Comp}_{\vec{c}}\ \vec{a}\times\vec{b} = \|\vec{a}\times\vec{b}\|.\cos\theta = \sqrt{14}\times 0.875 = 3.274 \quad 2$$

Q.5 Given planes:

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$P_1 : x - 5y + 3z = 11$ —(I) , $P_2 : -3x + 2y - 2z = -7$ —(II)

Normal vector of P_1: $\vec{n}_1 = [1,-5,3]$

" " " P_2: $\vec{n}_2 = [-3,2,-2]$

$\because \frac{1}{-3} \neq \frac{-5}{2} \neq \frac{3}{-2}$, i.e, components of $\vec{n}_1$ & $\vec{n}_2$ are not proportional. $\therefore \vec{n}_1$ & $\vec{n}_2$ are not $\parallel$. $\Rightarrow P_1 \nparallel P_2$. 2

For line of intersection of P_1 & P_2:

(I) $\Rightarrow$ $x - 5y = 11 - 3z$ —(iii)

(II) $\Rightarrow$ $3x - 2y = 7 - 2z$ —(IV)

$3(iii) - (IV) \Rightarrow -13y = 26 - 7z \Rightarrow y = -2 + \frac{7}{13}z$ —(V)

$2(III) - 5(IV) \Rightarrow -13x = -13 + 4z \Rightarrow x = 1 - \frac{4}{13}z$ —(VI)

Let $z = t$, then we have,

$l: x = 1 - \frac{4}{13}t,\ y = -2 + \frac{7}{13}t,\ z = t,\ t \in \mathbb{R}$ —(vii)

This is the line of intersection of P_1 & P_2. 4

Q.6 Given

4 Vector differential Equation: $\ddot{\vec{r}}(t) = \frac{t}{(1+t^2)^2}\hat{i} + \frac{1}{(1+t)^2}\hat{j} + 2^{3t}.\hat{k}$ —(I)

I Cds: $\dot{\vec{r}}(0) = \frac{1}{2}\hat{i} + 2\hat{j} - \frac{2}{3\cdot\ln 2}\hat{k}$ —(II)

& $\vec{r}(0) = 6\hat{i} + \hat{j}$ —(III)

Integrating (I) on both sides w.r.t. 't'

$$\int \ddot{\vec{r}}(t).\,dt = \int \frac{t}{((1+t^2)^2}.dt\;\hat{i} + \int \frac{dt}{(1+t)^2}\;\hat{j} + \int 2^{3t}.dt\;\hat{k}$$

$$\Rightarrow \dot{\vec{r}}(t) = \hat{i}\int (1+t^2)^{-2}.t.\,dt + \hat{j}\int (1+t)^{-2}.\,dt + \hat{k}\int 2^{3t}.\,dt$$

$$= -\frac{1}{2(1+t^2)}.\hat{i} - \frac{1}{1+t}.\hat{j} + \frac{2^{3t}}{3\ln 2}.\hat{k} + \vec{C} \quad \text{—(IV)}$$

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where $\vec{C}$ is arbitrary constt of integration

At $t=0$; $\dot{\vec{r}}(0) = -\frac{1}{2}\hat{i} - \hat{j} + \frac{1}{3\cdot\ln 2}\hat{k} + \vec{C}$

$$\Rightarrow \frac{1}{2}\hat{i} + 2\hat{j} - \frac{2}{3\ln 2}\hat{k} = -\frac{\hat{i}}{2} - \hat{j} + \frac{1}{3\ln 2}\hat{k} + \vec{C} \quad \text{using (II)}$$

$$\Rightarrow \vec{C} = \hat{i} + 3\hat{j} - \frac{1}{\ln 2}\hat{k} \quad 1$$

$$\therefore \text{(IV)} \Rightarrow \dot{\vec{r}}(t) = \left(-\frac{1}{2(1+t^2)} + 1\right)\hat{i} + \left(\frac{-1}{1+t} + 3\right)\hat{j} + \left(\frac{2^{3t}}{3\ln 2} - \frac{1}{\ln 2}\right)\hat{k}$$

Integrating again w.r.t. 't'

$$\int \dot{\vec{r}}(t).dt = \hat{i}\int \left[1 - \frac{1}{2(1+t^2)}\right]dt + \hat{j}\left[\int \left(3 - \frac{1}{1+t}\right)dt\right] + \hat{k}\int\left(\frac{2^{3t}}{3.\ln 2} - \frac{1}{\ln 2}\right)dt$$

$$\Rightarrow \vec{r}(t) = \hat{i}\left(t - \frac{\tan^{-1}t}{2}\right) + \hat{j}\left(3t - \ln(1+t)\right) + \hat{k}\left(\frac{2^{3t}}{9.(\ln 2)^2} - \frac{t}{\ln 2}\right) + \vec{d} \quad 1$$

where $\vec{d}$ is arbitrary constt of integration.

At $t=0$,

$$\vec{r}(0) = 0.\hat{i} + 0.\hat{j} + \frac{1}{9.(\ln 2)^2}\hat{k} + \vec{d}$$

$$\overset{\text{(III)}}{\Rightarrow} 6\hat{i} + \hat{j} = \frac{\hat{k}}{9(\ln 2)^2} + \vec{d} \Rightarrow \vec{d} = 6\hat{i} + \hat{j} - \frac{\hat{k}}{9(\ln 2)^2}$$

$$\therefore \vec{r}(t) = \hat{i}\left[6 + t - \frac{\tan^{-1}t}{2}\right] + \hat{j}\left[1 + 3t - \ln(1+t)\right] + \hat{k}\left[-\frac{1}{9(\ln 2)^2} - \frac{t}{\ln 2} + \frac{2^{3t}}{9(\ln 2)^2}\right] \quad 1$$