

King Saud University
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Department of Statistics and Operations
Research

STAT 223
Theory of Statistics 1

Exercises
1438/1439 H

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Chapter 1 Exercises: Introduction

- 1.1 Suppose that 4 out of 12 buildings in a certain city violate the building code. A building engineer randomly inspects a sample of 3 new buildings in the city.
- Find the probability distribution function of the random variable X representing the number of buildings that violate the building code in the sample.
 - Find the probability that
 - none of the buildings in the sample violating the building code.
 - one building in the sample violating the building code.
 - at least one building in the sample violating the building code.
 - Find the expected number of buildings in the sample that violate the building code.
 - Find $\text{Var}(X)$.
- 1.2 On average, a certain intersection results in 3 traffic accidents per day. Assuming Poisson distribution,
- what is the probability that at this intersection:
 - no accidents will occur in a given day?
 - More than 3 accidents will occur in a given day?
 - Exactly 5 accidents will occur in a period of two days?
 - what is the average number of traffic accidents in a period of 4 days?
- 1.3 If the random variable X has a uniform distribution on the interval $(0,10)$, then
- $P(X < 6)$ equals to
 - The mean of X is
 - The variance X is
- 1.4 Suppose that Z is distributed according to the standard normal distribution. Then,
- the area under the curve to the left of 1.43 is:
 - the area under the curve to the right of 0.89 is:
 - the area under the curve between 2.16 and 0.65 is:
 - the value of k such that $P(0.93 < Z < k) = 0.0427$ is:
- 1.5 The finished inside diameter of a piston ring is normally distributed with a mean of 12 centimeters and a standard deviation of 0.03 centimeter. Find,
- the proportion of rings that will have inside diameter less than 12.05 centimeters.
 - the proportion of rings that will have inside diameter exceeding 11.97 centimeters.
 - the probability that a piston ring will have an inside diameter between 11.95 and 12.05 centimeters.
- 1.6 Let X be $N(\mu, \sigma^2)$ so that $P(X < 89) = 0.90$ and $P(X < 94) = 0.95$. find μ and σ^2 .
- 1.7 Assume the length (in minutes) of a particular type of a telephone conversation is a random variable with a probability density function of the form:

$$f(x) = \begin{cases} 0.2 e^{-0.2x}; & x \geq 0 \\ 0; & \text{elsewhere} \end{cases}$$

Calculate:

- $P(3 < x < 10)$.
- The cdf of X .
- The mean and the variance of X .

- 1.8 Find the moment-generating function of X , if you know that $f(x) = 2e^{-2x}, x > 0$.
- 1.9 For a chi-squared distribution, find
- $\chi_{0.025}^2$ when $\nu = 15$.
 - $\chi_{0.01}^2$ when $\nu = 7$.
 - $\chi_{0.99}^2$ when $\nu = 22$.
- 1.10 If $(1 - 2t)^{-6}, t < \frac{1}{2}$, is the mgf of the random variable X , find $P(X < 5.23)$.
- 1.11 Find:
- $t_{0.95}$ when $\nu = 28$.
 - $t_{0.005}$ when $\nu = 16$.
 - $-t_{0.01}$ when $\nu = 4$.
 - $P(T > 1.318)$ when $\nu = 24$.
 - $P(-1.356 < T < 2.179)$ when $\nu = 12$.
- 1.12 If $f(x) = \theta x^{\theta-1}, 0 < x < 1$, find the distribution of $Y = -\ln X$.
- 1.13 If $f(x) = 1, 0 < x < 1$. Find the pdf of $Y = \sqrt{X}$.
- 1.14 If $X \sim \text{Uniform}(0,1)$, find the pdf of $Y = -2\ln X$. Name the distribution and its parameter values.
- 1.15 Suppose independent random variables X and Y are such that $M_{X+Y}(t) = \frac{e^{2t}-1}{2t-t^2}$. If $f(x) = \lambda e^{-\lambda x}, x > 0$, what is the distribution of Y .
- 1.16 Let X_1 and X_2 are two random variables have the joint probability distribution as follows:

		X_1	
		0	1
X_2	0	0.05	0.58
	1	0.3	0.07

- Prove that X_1 and X_2 are dependent random variables.
 - Find the pmf of the random variable $Y = 2X_1 + X_2$.
- 1.17 If $X \sim \text{Uniform}(0,1)$ independent of $Y \sim \text{Exponential}(1)$, find the distribution of $Z = X + Y$.
- 1.18 If $X_1 \sim \chi_n^2$ and $X_2 \sim \chi_m^2$ are independent random variables. Find the distribution of $Y = X_1 + X_2$.

Chapter 2 Exercises: Sampling Distribution

- 2.1 If e^{3t+4t^2} is the mgf of the random variable \bar{X} with sample size 6, find $P(-2 < \bar{X} < 6)$.
- 2.2 Let \bar{X} be the mean of a random sample of size 5 from a normal distribution with $\mu = 0$ and $\sigma^2 = 125$. Determine c so that $P(\bar{X} < c) = 0.975$.
- 2.3 Determine the mean and variance of the mean \bar{X} of a random sample of size 9 from a distribution having pdf $f(x) = 4x^3, 0 < x < 1$, zero elsewhere.
- 2.4 Let Z_1, Z_2, \dots, Z_{16} , be a random sample of size 16 from the standard normal distribution $N(0, 1)$. Let X_1, X_2, \dots, X_{64} be a random sample of size 64 from the normal distribution $N(\mu, 1)$. The two samples are independent.
- Find $P(Z_1 < 2)$.
 - Find $P(\sum_{i=1}^{16} Z_i > 2)$
 - Find $P(\sum_{i=1}^{16} Z_i^2 > 6.91)$
 - Let S^2 be the sample variance of the first sample. Find c such that $P(S^2 > c) = 0.05$.
 - What is the distribution of Y , where $Y = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64} (X_i - \mu)^2$
 - Find $E(Y)$.
 - Find $Var(Y)$.
 - Approximate $P(Y > 105)$.
 - Find c such that $c \frac{\sum_{i=1}^{16} Z_i^2}{Y} \sim F_{16,80}$
 - Let $Q \sim X_{60}^2$. Find c such that $P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95$
 - Find c such that $P(F_{60,20} > c) = 0.99$.
- 2.5 Let X be $N(5,10)$. Find $P(0.04 < (X - 5)^2 < 38.4)$.
- 2.6 Let S^2 be the variance of a random sample of size 6 from the normal distribution $N(\mu, 12)$. Find
- Mean and variance of S^2 .
 - Distribution of S^2 .
 - $P(2.30 < S^2 < 22.2)$.
- 2.7 Let X_1, X_2 and X_3 be iid random variable, each with pdf $f(x) = e^{-x}, 0 < x < \infty$; and let $Y_1 < Y_2 < Y_3$ be the order statistics of the random variables. Find:
- The distribution of $Y_1 = \text{minimum}(X_1, X_2, X_3)$.
 - $P(3 \leq Y_1)$.
 - The joint pdf of Y_2 and Y_3 .
- 2.8 Let $Y_1 < Y_2 < \dots < Y_n$ be the order statistics from a Weibull distribution. Find the distribution function and pdf of Y_1 .

Chapter 3 Exercises: Point Estimation

- 3.1 Suppose X_1, X_2, \dots, X_n is a random sample from gamma distribution:

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x > 0$$

Derive the MME for parameters α and β .

- 3.2 Find the MME and the MLE for the parameter p of Bernoulli distribution:

$$f(x; p) = p^x q^{1-x}, x = 0, 1.$$

Then, determine the unbiasedness, sufficiency and consistency of the MLE.

- 3.3 Let $f(x, \theta) = \theta e^{-\theta x}; x > 0$, and let T be an estimator for $\tau(\theta)$. Study if T is unbiased, consistent estimator for $\tau(\theta)$, then compute MSE in the three cases:

(a) $T = \bar{X}$ and $\tau(\theta) = \frac{1}{\theta}$.

(b) $T = \frac{1}{\bar{X}}$ and $\tau(\theta) = \theta$.

(c) $T = \frac{n-1}{\sum X_i}$ and $\tau(\theta) = \theta$.

- 3.4 If X_1, X_2, \dots, X_n be a random sample from $f(x; \theta)$. Show if the given statistic T is sufficient statistic for θ :

$$f(x; \theta) = e^{-(x-\theta)}, x > \theta; T = Y_1 = \text{Minimum}(X_1, X_2, \dots, X_n).$$

- 3.5 Suppose for a given random variable T_1 and T_2 be two independents unbiased estimators for θ and with the same variance σ^2 . Define two random variables as

$$Y = \frac{3T_1 + 2T_2}{5} \quad \text{and} \quad Z = \frac{T_1 + 2T_2}{3}$$

Find $MSE(Y)$ and $MSE(Z)$ and compare between them.

- 3.6 Let $f(x, \theta) = \frac{1}{\theta}; x \in (0, \theta)$, and let T be an estimator for θ . Study if T is unbiased, consistent and compute MSE, then compare between their variances for the following cases:

(a) $T = Y_1 = \text{Minimum}(X_1, X_2, \dots, X_n)$.

(b) $T = nY_1$.

(c) $T = 2\bar{X}$.

(d) $T = \frac{n+1}{n} Y_n$.

- 3.7 For a random sample X_1, X_2, \dots, X_n drawn from the following distributions, find the Fisher information, $I_X(\theta)$:

(a) *Bernoulli*(θ).

(b) *Exponential*($\frac{1}{\theta}$)

(c) $N(\theta, \sigma^2)$ when θ is unknown and σ^2 is known.

- 3.8 Let X_1, X_2, \dots, X_n be a random sample drawn $N(\mu, \sigma^2)$, σ^2 is known. Find:

(a) The CRLB for

i. $\tau(\mu) = \mu$ ii. $\tau(\mu) = e^\mu$ iii. $\tau(\mu) = \frac{1}{\mu+1}$

(b) The MVUE for μ .

- 3.10 Let X_1, X_2, \dots, X_n be a random sample from a distribution with pdf

$$f(x; \theta) = \theta^2 x e^{-x\theta}, \quad x > 0, \quad \theta > 0$$

- (a) Argue that $Y = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ .
- (b) Compute $E\left(\frac{1}{Y}\right)$ and find the function of Y which is the unique MVUE of θ .
- 3.11 Let X_1, X_2, \dots, X_n , $n > 2$, be a random sample from the binomial distribution $Binomial(1, \theta)$.
- (a) Show that $T_1 = X_1 + X_2 + \dots + X_n$ is a complete sufficient statistic for θ .
- (b) Find the MVUE of θ .
- (c) Let $T_2 = \frac{X_1 + X_2}{2}$ and prove that T_2 is an unbiased estimator for θ .

Chapter 4 Exercises: Interval Estimation

- 4.1 Let X_1, \dots, X_n be a random sample from the $\Gamma(2, \theta)$ distribution, where θ is unknown. Let $Y = \sum_{i=1}^n X_i$.
- Find the distribution of Y and determine c so that cY is an unbiased estimator of θ .
 - If $n = 5$, show that $P\left(9.59 < \frac{2Y}{\theta} < 34.2\right) = 0.95$.
 - Using Part (b), show that if y is the value of Y once the sample is drawn, then the interval $\left(\frac{2y}{34.2}, \frac{2y}{9.59}\right)$ is a 95% confidence interval for θ .
 - Suppose the sample results in the values,
44.8079 1.5215 12.1929 12.5734 43.2305
Based on these data, obtain the point estimate of θ as determined in Part (a) and the computed 95% confidence interval in Part (c). What does the confidence interval mean?
- 4.2 Let the observed value of the mean \bar{X} of a random sample of size 20 from a distribution that $N(\mu, 80)$ be 81.2. Find a 95% confidence interval for μ .
- 4.3 Let \bar{X} be the mean of a random sample of size n from a distribution that is $N(\mu, 9)$. Find n such that $P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.90$, approximately.
- 4.4 Let a random sample of size 17 from the normal distribution $N(\mu, \sigma^2)$ yield $\bar{x} = 4.7$ and $s^2 = 5.76$. Determine a 90% confidence interval for μ .
- 4.5 Let \bar{x} be the observed mean of a random sample of size n from a distribution having mean μ and known variance σ^2 . Find n so that $\bar{x} - \frac{\sigma}{4}$ to $\bar{x} + \frac{\sigma}{4}$ is an approximate 95% confidence interval for μ .
- 4.6 Let X_1, \dots, X_n be a random sample from gamma distributions with known parameter $\alpha = 3$ and unknown $\beta > 0$. Discuss the construction of a confidence interval for β .
Hint: What is the distribution of $2 \sum_{i=1}^n \frac{X_i}{\beta}$?
- 4.7 If 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, 7.5 are the observed values of a random sample of size 9 from a distribution that is $N(8, \sigma^2)$, construct a 90% confidence interval for σ^2 .
- 4.8 The breaking strengths in pounds of five specimens of manila rope of diameter $\frac{3}{16}$ inch were found to be 660, 460, 540, 580 and 550.
- Estimate the mean breaking strength by a 95% confidence interval assuming normality.
 - Estimate σ^2 by a 90% confidence interval; also σ .
- 4.9 Let \bar{X} denote the mean of a random sample of size 25 from a gamma distribution with $\alpha = 4$ and $\beta > 0$. Use the Central Limit Theorem to find an approximate 0.954 confidence interval for μ , the mean of the gamma distribution. Hint: Use the random variable $\frac{(\bar{X}-4\beta)}{\sqrt{4\beta/25}} = \frac{5\bar{X}}{2\beta} - 10$.
- 4.10 Let Y be $Bin(300, p)$. If the observed value of Y is $y = 75$, find an approximate 90% confidence interval for p .

- 4.11 Let \bar{X} be the mean of a random sample from the exponential distribution, $Exp(\theta)$.
- Show that \bar{X} is an unbiased point estimator of θ .
 - Using the MGF technique determine the distribution of \bar{X} .
 - Use (b) to show that $Y = 2n\bar{X}/\theta$ has a χ^2 distribution with $2n$ degrees of freedom.
 - Based on Part (c), find a 95% confidence interval for θ if $n = 10$. Hint: Find c and d such that $P\left(c < \frac{2n\bar{X}}{\theta} < d\right) = 0.95$ and solve the inequalities for θ .
- 4.12 Let X_1, \dots, X_n be a random sample from $N(\mu, \sigma^2)$, where both parameters μ and σ^2 are unknown. If $n = 9$ and $s^2 = 7.93$, find a 95% confidence interval for σ^2 .
- 4.13 When 100 tacks were thrown on a table, 60 of them landed point up. Obtain a 95% confidence interval for the probability that a tack of this type will land point up. Assume independence.
- 4.14 Find a 90 percent confidence interval for the mean of a normal distribution with $\sigma = 3$ given the sample (3.3, -0.3, -0.6, -0.9). What would be the confidence interval if σ were unknown?
- 4.15 A sample was drawn from each of five populations assumed to be normal with the same variance. The values of $(n - 1)S^2 = \sum(X_i - \bar{X})^2$ and n , the sample size, were
- | | | | | |
|---------|----|----|----|----|
| S^2 : | 40 | 30 | 20 | 42 |
| n : | 6 | 4 | 3 | 7 |
- Find 98% confidence interval for the common variance.
- 4.16 Let S_1^2 and S_2^2 denote, respectively, the variances of random samples, of sizes n and m , from two independent distributions that are $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$.
- Derive $(1 - \alpha)100\%$ confidence interval for the common unknown variance σ^2 .
 - Find 99% confidence interval for σ^2 if $n = 12, s_1^2 = 4.74, m = 15, s_2^2 = 5.66$.

Chapter 5 Exercises: Bayesian Estimation

- 5.1 Let X_1, X_2, \dots, X_n be a random sample from Bernoulli with parameter θ , and the prior distribution of Θ is a uniform distribution, where $0 < \theta < 1$. Find the posterior distribution and the Bayes' point estimator of Θ when the loss function be the squared error loss function.
- 5.2 Let Y have a binomial distribution in which $n = 20$ and $p = \theta$. The prior probability on Θ is $Beta(a, b)$, where $a, b > 0$ are known constants. Find the following:
- Posterior distribution.
 - Bayes' point estimate of Θ , when $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$.
- 5.3 Let X_1, X_2, \dots, X_{10} denote a random sample from a Poisson distribution with mean θ , $0 < \theta < \infty$. Let $Y = \sum_{i=1}^{10} X_i$. Use the loss function to be $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If Θ has the pdf $h(\theta) = \frac{\theta^2 e^{-\frac{1}{2}\theta}}{16}$, for $0 < \theta < \infty$. Find:
- The posterior distribution.
 - The Bayes' solution $\delta(y)$ for a point estimate for θ , when $Y = 22$.
 - The 90% credible interval for Θ , when $Y = 22$.
- 5.4 Let Y_n be the n th order statistic of a random sample of size n from a distribution with pdf $f(x|\theta) = \frac{1}{\theta}$, $0 < x \leq \theta$, zero elsewhere. Take the loss function to be $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y_n)]^2$. Let θ be an observed value of the random variable Θ , which has pdf $h(\theta) = \frac{\beta \alpha^\beta}{\theta^{\beta+1}}$, $\alpha < \theta < \infty$, zero elsewhere, with $\alpha > 0, \beta > 0$. Find the Bayes' solution $\delta(y_n)$ for a point estimate of θ .
- 5.5 In Exercise 5.5, let $n = 4$ from the uniform pdf $f(x, \theta) = \frac{1}{\theta}$, $0 < x < \theta$, and the prior pdf be $g(\theta) = \frac{2}{\theta^3}$, $1 < \theta < \infty$, zero elsewhere. Find:
- The Bayesian estimator $\delta(Y_4)$ of θ , based upon the sufficient statistic Y_4 , using the loss function $[\theta - \delta(y_4)]^2$.
 - The Bayesian estimator $\delta(Y_4)$ of θ , based upon the sufficient statistic Y_4 , using the loss function $|\delta(y_4) - \theta|$.
- 5.6 Consider the model

$$X_i|\theta \sim \text{iid Exponential}\left(\frac{1}{\theta}\right)$$

$$\Theta \sim \text{Gamma}\left(\alpha, \frac{1}{\beta}\right)$$

Find the following:

- Posterior distribution of Θ .
- Bayes' point estimate of Θ , use $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$.
- 95% creditable interval for Θ .
- If $X_1 = 2.5, X_2 = 3.61, X_3 = 4.8, X_4 = 2.74, X_5 = 3.95$ and $\alpha = 2, \beta = 4$. Calculate (b) and (c).