Coordinate System and Change of Bases

Definition

If $S = \{v_1, \ldots, v_n\}$ is a basis of the vector space V and if $v \in V$ such that

$$v = x_1 v_1 + \ldots x_n v_n$$

then $(x_1, \ldots x_n)$ are called the system of coordinates of the vector v in the basis S. We denote $[v]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Theorem

If $B = \{v_1, \ldots, v_n\}$ and $C = \{u_1, \ldots, u_n\}$ are two bases of the vector space V. The matrix ${}_CP_B \in M_n(\mathbb{R})$ with columns $[v_1]_C, \ldots, [v_n]_C$ is called the change of bases matrix from the basis B to the basis C. This matrix ${}_CP_B$ is invertible, ${}_CP_B^{-1} = {}_BP_C$ and

$$[v]_C = {}_CP_B[v]_B, \quad \text{for all } v \in V.$$

(${}_BP_C$ is the change of bases matrix from the basis C to the basis B.)

Note: The notation used in the book for the above definition is:

Transition Matrices The matrix P in Equation (7) is called the ***transition matrix*** from B' to B. For emphasis, we will often denote it by $P_{B'\to B}$. It follows from (8) that this matrix can be expressed in terms of its column vectors as

$$P_{B'\to B} = \left[[\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \cdots \mid [\mathbf{u}'_n]_B\right] \quad (9)$$

Similarly, the transition matrix from B to B' can be expressed in terms of its column vectors as

$$P_{B\to B'} = \left[[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid \cdots \mid [\mathbf{u}_n]_{B'}\right] \quad (10)$$

Remark There is a simple way to remember both of these formulas using the terms "old basis" and "new basis" defined earlier in this section: In Formula (9) the old basis is B' and the new basis is B, whereas in Formula (10) the old basis is B and the new basis is B'. Thus, both formulas can be restated as follows:

The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.

An Efficient Method for Computing Transition Matrices for R^n

Our next objective is to develop an efficient procedure for computing transition matrices *between bases for* R^n. As illustrated in Example 1, the first step in computing a transition matrix is to express each new basis vector as a linear combination of the old basis vectors. For R^n this involves solving n linear systems of n equations in n unknowns, each of which has the same coefficient matrix (why?). An efficient way to do this is by the method illustrated in Example 2 of Section 1.6, which is as follows:

A Procedure for Computing $P_{B \to B'}$

Step 1. Form the matrix $[B' \mid B]$.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be $[I \mid P_{B \to B'}]$.

Step 4. Extract the matrix $P_{B \to B'}$ from the right side of the matrix in Step 3.

This procedure is captured in the following diagram.

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}] \tag{14}$$

Example

Let $B = \{v_1 = (0,1,1), v_2 = (1,0,-2), v_3 = (1,1,0)\}$ be a basis of the vector space $\mathbb{R}^3$ and let $C = \{u_1 = (1,0,0), u_2 = (0,1,0), u_3 = (0,0,1)\}$ be the standard basis of $\mathbb{R}^3$.

We have ${}_CP_B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}$ and ${}_BP_C = \begin{pmatrix} -2 & 2 & -1 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$. If $[v]_C = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, then $[v]_B = {}_BP_C[v]_C = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Consider B and C two bases of vector space V such that the matrix ${}_CP_B = \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}$. Let u be a vector in V such that $[u]_C = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

then $[u]_B = {}_BP_C[u]_C = {}_CP_B^{-1}[u]_C = \frac{1}{3}\begin{pmatrix} 6y+3z \\ -x+z \\ -x+3y+z \end{pmatrix}$.

4.7 Row Space, Column Space, and Null Space

DEFINITION 1 For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}] \\ &\vdots \\ \mathbf{r}_m &= [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}] \end{aligned}$$

in R^n that are formed from the rows of A are called the ***row vectors*** of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m formed from the columns of A are called the ***column vectors*** of A.

EXAMPLE 1 Row and Column Vectors of a 2 x 3 Matrix

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The row vectors of A are

$$\mathbf{r}_1 = [2 \quad 1 \quad 0] \quad \text{and} \quad \mathbf{r}_2 = [3 \quad -1 \quad 4]$$

and the column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

The following definition defines three important vector spaces associated with a matrix.

DEFINITION 2 If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the ***row space*** of A, and the subspace of R^m spanned by the column vectors of A is called the ***column space*** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is called the ***null space*** of A.

We will sometimes denote the row space of A, the column space of A, and the null space of A by $\text{row}(A)$, $\text{col}(A)$, and $\text{null}(A)$, respectively.

THEOREM 4.7.4 *Elementary row operations do not change the row space of a matrix.*

THEOREM 4.7.5 *If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.*

THEOREM 4.7.6 *If A and B are row equivalent matrices, then:*

(*a*) *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.*

(*b*) *A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.*

▶ **EXAMPLE 6 Basis for a Row Space by Row Reduction**

Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solution Since elementary row operations do not change the row space of a matrix, we can find a basis for the row space of A by finding a basis for the row space of any row echelon form of A. Reducing A to row echelon form, we obtain (verify)

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 4.7.5, the nonzero row vectors of R form a basis for the row space of R and hence form a basis for the row space of A. These basis vectors are

$$\mathbf{r}_1 = [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4]$$
$$\mathbf{r}_2 = [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6]$$
$$\mathbf{r}_3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5]$$ ◀

EXAMPLE 7 Basis for a Column Space by Row Reduction

Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

that consists of column vectors of A.

Solution We observed in Example 6 that the matrix

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a row echelon form of A. Keeping in mind that A and R can have different column spaces, we cannot find a basis for the column space of A directly from the column vectors of R. However, it follows from Theorem 4.7.6(b) that if we can find a set of column vectors of R that forms a basis for the column space of R, then the *corresponding* column vectors of A will form a basis for the column space of A.

Since the first, third, and fifth columns of R contain the leading 1's of the row vectors, the vectors

$$\mathbf{c}_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R. Thus, the corresponding column vectors of A, which are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A.

If $S = \{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ is a set of vectors and F the vector sub-space generated by S. We have the following two algorithms to construct a basis of F.

First Algorithm

1. Construct the matrix A such that its rows are the vectors of S
2. The non zeros rows of any row echelon form of the matrix A are a basis of the vector space $F = \langle S \rangle$.

Second Algorithm

1. Construct the matrix A such that its columns are the vectors of S
2. Take any row echelon form C of the matrix A.
3. Let $C_{k_1}, \ldots C_{k_p}$ be the columns which contain a leading number and $k_1 < \ldots < k_p$. Then $\{v_{k_1}, \ldots, v_{k_p}\}$ is a basis of the vector space $F = \langle S \rangle$.

Ex. Find a basis for the subspace $W \subseteq \mathbb{R}^5$, generated by $(1,-1,2,0,-1)$, $(2,-1,-2,0,1)$, $(-1,0,4,0,-2)$, and $(0,-1,6,0,-3)$.

Solution:

$$\begin{bmatrix} 1 & -1 & 2 & 0 & -1 \\ 2 & -1 & -2 & 0 & 1 \\ -1 & 0 & 4 & 0 & -2 \\ 0 & -1 & 6 & 0 & -3 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & -1 \\ 0 & 1 & -6 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the subspace W is

$B = \{(1,-1,2,0,-1), (0,1,-6,0,3)\}$

Q. What is the dimension of the subspace W?

Q3: Let V be the subspace of $\mathbb{R}^3$ **spanned** by the set S={ v_1=(1, 2,3), v_2=(2, 4,6), v_3=(4, 6, 6)}. Find a **subset** of S that forms a basis of V.

Answer:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 6 \\ 3 & 6 & 6 \end{bmatrix} \xrightarrow[-3R_{13}]{-2R_{12}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow{6R_{23}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since columns 1 and 3 have leading ones, then v_1 and v_3 forms a basis of V.

Homework

17. Find a basis for the subspace of R^3 that is spanned by the vectors

$$\mathbf{v}_1 = (1, 0, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (2, 0, 1), \quad \mathbf{v}_4 = (0, 0, -1)$$

18. Find a basis for the subspace of R^4 that is spanned by the vectors

$$\mathbf{v}_1 = (1, 1, 1, 1), \quad \mathbf{v}_2 = (2, 2, 2, 0), \quad \mathbf{v}_3 = (0, 0, 0, 3),$$
$$\mathbf{v}_4 = (3, 3, 3, 4)$$

4.8 Rank, Nullity

THEOREM 4.8.1 *The row space and the column space of a matrix A have the same dimension.*

Rank and Nullity The dimensions of the row space, column space, and null space of a matrix are such important numbers that there is some notation and terminology associated with them.

DEFINITION 1 The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by $\text{rank}(A)$; the dimension of the null space of A is called the ***nullity*** of A and is denoted by $\text{nullity}(A)$.

EXAMPLE 1 Rank and Nullity of a 4 x 6 Matrix

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

(verify). Since this matrix has two leading 1's, its row and column spaces are two-dimensional and $\text{rank}(A) = 2$. To find the nullity of A, we must find the dimension of the solution space of the linear system $A\mathbf{x} = \mathbf{0}$. This system can be solved by reducing its augmented matrix to reduced row echelon form. The resulting matrix will be identical to (1), except that it will have an additional last column of zeros, and hence the corresponding system of equations will be

$$\begin{aligned} x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 &= 0 \\ x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 &= 0 \end{aligned}$$

Solving these equations for the leading variables yields

$$\begin{aligned} x_1 &= 4x_3 + 28x_4 + 37x_5 - 13x_6 \\ x_2 &= 2x_3 + 12x_4 + 16x_5 - 5x_6 \end{aligned} \tag{2}$$

from which we obtain the general solution

$$\begin{aligned} x_1 &= 4r + 28s + 37t - 13u \\ x_2 &= 2r + 12s + 16t - 5u \\ x_3 &= r \\ x_4 &= s \\ x_5 &= t \\ x_6 &= u \end{aligned}$$

or in column vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

Because the four vectors on the right side of (3) form a basis for the solution space, $\text{nullity}(A) = 4$.

THEOREM 4.8.2 Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$rank(A) + nullity(A) = n \tag{4}$$

EXAMPLE 3 The Sum of Rank and Nullity

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns, so

$$\text{rank}(A) + \text{nullity}(A) = 6$$

This is consistent with Example 1, where we showed that

$$\text{rank}(A) = 2 \quad \text{and} \quad \text{nullity}(A) = 4$$

THEOREM 4.8.5 *If A is any matrix, then* $\text{rank}(A) = \text{rank}(A^T)$.

Proof

$$\text{rank}(A) = \dim(\text{row space of } A) = \dim(\text{column space of } A^T) = \text{rank}(A^T). \blacktriangleleft$$

This result has some important implications. For example, if A is an $m \times n$ matrix, then applying Formula (4) to the matrix A^T and using the fact that this matrix has m columns yields

$$\text{rank}(A^T) + \text{nullity}(A^T) = m$$

which, by virtue of Theorem 4.8.5, can be rewritten as

$$\text{rank}(A) + \text{nullity}(A^T) = m \tag{5}$$

THEOREM 4.8.8 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is I_n.*

(*d*) *A is expressible as a product of elementary matrices.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

(*h*) *The column vectors of A are linearly independent.*

(*i*) *The row vectors of A are linearly independent.*

(*j*) *The column vectors of A span R^n.*

(*k*) *The row vectors of A span R^n.*

(*l*) *The column vectors of A form a basis for R^n.*

(*m*) *The row vectors of A form a basis for R^n.*

(*n*) *A has rank n.*

(*o*) *A has nullity* 0.