

The following theorem establishes a relationship between the inverse of a matrix and the inverse of its transpose.

**THEOREM 1.4.9** *If  $A$  is an invertible matrix, then  $A^T$  is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

## 1.5 Elementary Matrices and a Method for Finding $A^{-1}$

**DEFINITION 2** A matrix  $E$  is called an *elementary matrix* if it can be obtained from an identity matrix by performing a *single* elementary row operation.

### ▶ EXAMPLE 1 Elementary Matrices and Row Operations

Listed below are four elementary matrices and the operations that produce them.

$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
Multiply the second row of $I_2$ by $-3$ .	Interchange the second and fourth rows of $I_4$ .	Add 3 times the third row of $I_3$ to the first row.	Multiply the first row of $I_3$ by 1. ▶

**THEOREM 1.5.1 Row Operations by Matrix Multiplication**

*If the elementary matrix  $E$  results from performing a certain row operation on  $I_m$  and if  $A$  is an  $m \times n$  matrix, then the product  $EA$  is the matrix that results when this same row operation is performed on  $A$ .*

**▶ EXAMPLE 2 Using Elementary Matrices**

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results from adding 3 times the first row of  $I_3$  to the third row. The product  $EA$  is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is precisely the matrix that results when we add 3 times the first row of  $A$  to the third row. ◀

**THEOREM 1.5.3 Equivalent Statements**

*If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent, that is, all true or all false.*

- (a)  *$A$  is invertible.*
- (b) *The reduced row echelon form of  $A$  is  $I_n$ .*
- (c)  *$A$  is expressible as a product of elementary matrices.*

We know from the discussion at the beginning of this section that if  $E$  is an elementary matrix that results from performing an elementary row operation on an identity matrix  $I$ , then there is a second elementary row operation, which when applied to  $E$  produces  $I$  back again. Table 1 lists these operations. The operations on the right side of the table are called the *inverse operations* of the corresponding operations on the left.

Table 1

Row Operation on $I$ That Produces $E$	Row Operation on $E$ That Reproduces $I$
Multiply row $i$ by $c \neq 0$	Multiply row $i$ by $1/c$
Interchange rows $i$ and $j$	Interchange rows $i$ and $j$
Add $c$ times row $i$ to row $j$	Add $-c$ times row $i$ to row $j$

► **EXAMPLE 3 Row Operations and Inverse Row Operations**

In each of the following, an elementary row operation is applied to the  $2 \times 2$  identity matrix to obtain an elementary matrix  $E$ , then  $E$  is restored to the identity matrix by applying the inverse row operation.

$$\begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \uparrow \qquad \qquad \qquad \uparrow \\ \text{Multiply the second} \quad \text{Multiply the second} \\ \text{row by } 7. \qquad \qquad \text{row by } \frac{1}{7}. \end{array}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \uparrow \qquad \qquad \qquad \uparrow \\ \text{Interchange the first} \quad \text{Interchange the first} \\ \text{and second rows.} \quad \text{and second rows.} \end{array}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \uparrow \qquad \qquad \qquad \uparrow \\ \text{Add 5 times the} \quad \text{Add } -5 \text{ times the} \\ \text{second row to the} \quad \text{second row to the} \\ \text{first.} \qquad \qquad \text{first.} \end{array} \quad \blacktriangleleft$$

*A Method for Inverting  
Matrices*

As a first application of Theorem 1.5.3, we will develop a procedure (or algorithm) that can be used to tell whether a given matrix is invertible, and if so, produce its inverse. To derive this algorithm, assume for the moment, that  $A$  is an invertible  $n \times n$  matrix. In Equation (3), the elementary matrices execute a sequence of row operations that reduce  $A$  to  $I_n$ . If we multiply both sides of this equation on the right by  $A^{-1}$  and simplify, we obtain

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

But this equation tells us that *the same sequence of row operations that reduces  $A$  to  $I_n$  will transform  $I_n$  to  $A^{-1}$* . Thus, we have established the following result.

**Inversion Algorithm** To find the inverse of an invertible matrix  $A$ , find a sequence of elementary row operations that reduces  $A$  to the identity and then perform that same sequence of operations on  $I_n$  to obtain  $A^{-1}$ .

► **EXAMPLE 4 Using Row Operations to Find  $A^{-1}$**

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

**Solution** We want to reduce  $A$  to the identity matrix by row operations and simultaneously apply these operations to  $I$  to produce  $A^{-1}$ . To accomplish this we will adjoin the identity matrix to the right side of  $A$ , thereby producing a partitioned matrix of the form

$$[A \mid I]$$

Then we will apply row operations to this matrix until the left side is reduced to  $I$ ; these operations will convert the right side to  $A^{-1}$ , so the final matrix will have the form

$$[I \mid A^{-1}]$$

The computations are as follows:

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right]$$

← We added  $-2$  times the first row to the second and  $-1$  times the first row to the third.

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right]$$

← We added 2 times the second row to the third.

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

← We multiplied the third row by  $-1$ .

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

← We added 3 times the third row to the second and  $-3$  times the third row to the first.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

← We added  $-2$  times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \leftarrow$$

Often it will not be known in advance if a given  $n \times n$  matrix  $A$  is invertible. However, if it is not, then by parts (a) and (c) of Theorem 1.5.3 it will be impossible to reduce  $A$  to  $I_n$  by elementary row operations. This will be signaled by a row of zeros appearing on the *left side* of the partition at some stage of the inversion algorithm. If this occurs, then you can stop the computations and conclude that  $A$  is not invertible.

► **EXAMPLE 5 Showing That a Matrix Is Not Invertible**

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Applying the procedure of Example 4 yields

$$\left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array} \right]$$

← We added  $-2$  times the first row to the second and added the first row to the third.

$$\left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array} \right]$$

← We added the second row to the third.

Since we have obtained a row of zeros on the left side,  $A$  is not invertible.



*Properties of Invertible  
Matrices*

**THEOREM 1.6.3** *Let  $A$  be a square matrix.*

- (a) *If  $B$  is a square matrix satisfying  $BA = I$ , then  $B = A^{-1}$ .*
- (b) *If  $B$  is a square matrix satisfying  $AB = I$ , then  $B = A^{-1}$ .*

**THEOREM 1.6.4 Equivalent Statements**

*If  $A$  is an  $n \times n$  matrix, then the following are equivalent.*

- (a)  *$A$  is invertible.*
- (c) *The reduced row echelon form of  $A$  is  $I_n$ .*
- (d)  *$A$  is expressible as a product of elementary matrices.*

**THEOREM 1.6.5** *Let  $A$  and  $B$  be square matrices of the same size. If  $AB$  is invertible, then  $A$  and  $B$  must also be invertible.*

## 1.7 Diagonal, Triangular, and Symmetric Matrices

In this section we will discuss matrices that have various special forms. These matrices arise in a wide variety of applications and will play an important role in our subsequent work.

*Diagonal Matrices* A square matrix in which all the entries off the main diagonal are zero is called a *diagonal matrix*. Here are some examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A general  $n \times n$  diagonal matrix  $D$  can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \quad (1)$$

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; in this case the inverse of (1) is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix} \quad (2)$$

Powers of diagonal matrices are easy to compute; we leave it for you to verify that if  $D$  is the diagonal matrix (1) and  $k$  is a positive integer, then

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix} \quad (3)$$

► **EXAMPLE 1 Inverses and Powers of Diagonal Matrices**

If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

then

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}, \quad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$



Matrix products that involve diagonal factors are especially easy to compute. For example,

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} & d_1 a_{14} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} & d_2 a_{24} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} & d_3 a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \\ d_1 a_{41} & d_2 a_{42} & d_3 a_{43} \end{bmatrix}$$

In words, to multiply a matrix  $A$  on the left by a diagonal matrix  $D$ , multiply successive rows of  $A$  by the successive diagonal entries of  $D$ , and to multiply  $A$  on the right by  $D$ , multiply successive columns of  $A$  by the successive diagonal entries of  $D$ .

### Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called *lower triangular*, and a square matrix in which all the entries below the main diagonal are zero is called *upper triangular*. A matrix that is either upper triangular or lower triangular is called *triangular*.

#### ▶ EXAMPLE 2 Upper and Lower Triangular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

A general  $4 \times 4$  upper triangular matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general  $4 \times 4$  lower triangular matrix

**Remark** Observe that diagonal matrices are both upper triangular and lower triangular since they have zeros below and above the main diagonal. Observe also that a *square* matrix in row echelon form is upper triangular since it has zeros below the main diagonal.

**THEOREM 1.7.1**

- (a) *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*
- (b) *The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.*
- (c) *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*
- (d) *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*

**▶ EXAMPLE 3 Computations with Triangular Matrices**

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows from part (c) of Theorem 1.7.1 that the matrix  $A$  is invertible but the matrix  $B$  is not. Moreover, the theorem also tells us that  $A^{-1}$ ,  $AB$ , and  $BA$  must be upper triangular. We leave it for you to confirm these three statements by showing that

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix} \leftarrow$$