King Saud University
College of Science
Department of Mathematics

244 Math
Final Exam
Three Hours

Name:	**Student No.:**
Section / Group No.:	**Sequence No.:**

Question No.	**I**	**II**	**III**	**IV**	**V**	**VI**	**VII**	**VIII**	**Total**
Mark									

I. Determine whether the following statements are always true or sometimes false, and justify your answer with a logical argument or a counter example:

(a) If S_1 and S_2 are two sets of vectors in a vector space V with $span(S_1) = span(S_2)$ then $S_1 = S_2$.

☐ True ☐ False

Justification:

(b) If $\boldsymbol{v}$ is any non-zero vector in $\mathbb{R}^n$ then $\{0, \boldsymbol{v}\}$ forms a subspace of $\mathbb{R}^n$.

☐ True ☐ False

Justification:

(c) If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a linearly independent set of vectors then the set $\{k\boldsymbol{v}_1, k\boldsymbol{v}_2, k\boldsymbol{v}_3\}$ is also linearly independent for every non-zero scalar k.

☐ True ☐ False

Justification:

(d) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors in $\mathbb{R}^n$ then $\boldsymbol{u} \cdot \boldsymbol{v} \in \mathbb{R}^n$.

☐ True ☐ False

Justification:

(e) If E is any elementary matrix then A and EA have the same row space.

☐ True ☐ False

Justification:

(f) A basis of the row space of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -2 & 1 & 5 \\ -6 & 2 & -4 \end{bmatrix}$ is $\{(3,-1,2),(-2,1,5),(-6,2,-4)\}$.

☐ True ☐ False

Justification:

(g) If $T:\mathbb{R}^n \to \mathbb{R}^n$ is a linear operator and $Tx = 2x$ for some vector x in $\mathbb{R}^n$, then $\lambda = 2$ is an eigenvalue of T .

☐ True ☐ False

Justification:

(h) The only subspaces of $\mathbb{R}^n$ are $\{0\}$, lines through the origin, planes through the origin, and $\mathbb{R}^n$.

☐ True ☐ False

Justification:

(i) $\{(2,4,7),(8,16,14)\}$ is a linearly dependent set.

☐ True ☐ False

Justification:

(j) For any invertible matrix A, $rank\,(A) = rank\,(A^{-1})$.

☐ True ☐ False

Justification:

II. Choose the correct answer:

(a) If T is a contraction by a factor of $\frac{1}{5}$ in $\mathbb{R}^3$ then T^{-1} is a dilation by a factor of:

i. $-\frac{1}{5}$. ii. 5. iii. -5. iv. $\frac{1}{25}$.

(b) The system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ x & - & y & - & z & = & 5 \end{array}$$

has:

i. No solutions. ii. One solution. iii. Two solutions. iv. Infinitely many solutions.

(c) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two orthogonal vectors in $\mathbb{R}^n$ such that $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$ then $d(\boldsymbol{u}, \boldsymbol{v})$ is equal to:

i. 2. ii. 0. iii. $\sqrt{2}$. iv. 4.

(d) Let $A = \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & -6 & 4 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}$, then the dimension of the column space is:

i. 1. ii. 2. iii. 3. iv. 4.

(e) If λ is an eigenvalue of the invertible matrix A with respect to a non-zero vector x, then an eigenvalue of $\left(A^{-1}\right)^2$ is:

i. $\frac{1}{\lambda}$. ii. λ. iii. $\frac{1}{\lambda^2}$. iv. λ^2.

(f) The eigenvalues of the matrix $\begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -8 \\ 5 & 0 & 2 \end{bmatrix}$ are:

i. $1, -1, 2$. ii. $1, -1, 12$. iii. $1, -1, 5$. iv. $1, -1, 8$.

(g) The diagonal form of the previous matrix is:

i. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. ii. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 12 \end{bmatrix}$. iii. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. iv. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$.

(h) If A is a square 4×4 matrix of rank 3 then $nullity\,(3A)$ equals:

i. 9. ii. 3. iii. 12. iv. 1.

(i) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & s+3 \end{bmatrix}$. For what values of r and s will $rank\,(A)$ be equal to 2.

i. $s = 1, r = 2$. ii. $s = 1, r = -2$. iii. $s = -3, r = 2$. iv. $s = -3, r = -2$.

(j) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors in $\mathbb{R}^n$ with $\|\boldsymbol{u}\| = 1$, $\|\boldsymbol{v}\| = 2$ and $\|\boldsymbol{u} - \boldsymbol{v}\| = 5$ then $\boldsymbol{u} \cdot \boldsymbol{v}$ is equal to:

i. 8. ii. -20. iii. -5. iv. 5.

III. Let A be the matrix given by

$$\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}.$$

a) Find the eigenvalues, the eigenvectors corresponding to the eigenvalues, and a basis for the eigenspace of A .

b) Find a matrix P that diagonalizes A . What is $P^{-1}AP$.

IV. Determine whether multiplication by A is a one-to-one linear transformation where

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}.$$

V. If A is the matrix given by

$$\begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}$$

a) Find the rank and nullity of A .

b) Find a basis for the nullspace of A .

c) Determine whether the solution space is the origin only, a line through the origin, a plane through the origin, or all of $\mathbb{R}^3$.

VI. Find a subset of the vectors $\{v_1, v_2, v_3, v_4, v_5\}$ that forms a basis for the space spanned by the vectors, then express each vector not in the basis as a linear combination of the basis vectors, where $v_1 = (1,-1,5,2), v_2 = (-2,3,1,0), v_3 = (4,-5,9,4), v_4 = (0,4,2,-3)$, and $v_5 = (-7,18,2,-8)$.

VII. Find a matrix K such that $AKB = C$ given that

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix}.$$

BONUS QUESTION:(حل هذا السؤال اختيارى)

If $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ then the cross product $\boldsymbol{u} \times \boldsymbol{v}$ is the vector defined by

$$\boldsymbol{u} \times \boldsymbol{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

(a) Show that $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0$.

(b) Use part (a) to show that $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) \neq (\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$ in general.

(c) Show that $\boldsymbol{u} \perp (\boldsymbol{u} \times \boldsymbol{v})$.

Good Luck ☺