King Saud University, College of Science, Department of Mathematics

Math-280 (Introduction to Real Analysis)

Second Midterm Exam [Time: 90 Minutes]/ 1^{st} Semester, 1436- -1437 H.

Exercise 1 [2+3+3=8 Marks]:

1. Calculate the following limit: $\lim\limits_{x\to\infty} \dfrac{x\sin x}{x^2+5}$.

2. Calculate the following limit: $\lim\limits_{h\to 0} \dfrac{\sqrt{a+h}-\sqrt{a}}{h}$. What does this limit represent?

3. Using the $\varepsilon-\delta$–definition of the limit, show "*rigorously*" that $\lim\limits_{x\to 0} \dfrac{x}{1+\sin^2 x} = 0$.

Exercise 2 [2+3+3=8 Marks]:

1. Find the value of the constant c that makes f continuous at $x = 0$:

$$f(x) = \begin{cases} \dfrac{\tan^2 x + 2x}{x+x^2}, & x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \setminus \{-1, 0\}, \\ c & , x = 0 \end{cases}$$

2. Show that the equation $x2^x = 1$ has a solution in $(0, 1)$.

3. Show that the function $g(x) = e^{-x}$ is uniformly continuous on $[0, \infty)$.

Exercise 3 [3+3+3=9 Marks]:

1. Sketch the graph of the function $f(x) = x^4 - 2x^2$. Show all of your work.

2. Show that if $\dfrac{a_0}{1} + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \ldots + \dfrac{a_n}{n+1} = 0$ then $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0$ for some x in $[0, 1]$.

3. Among all rectangles of perimeter $L = 16\ cm$ find the one with largest area. What is the area of this rectangle with largest area? What is the characteristic of this rectangle?

.... Good Luck ....

Typical Answers to Second midterm exam, Math 280
2015/2016, 1436/1437 H

Exercise 1:-

1)
$$\lim_{x\to\infty}\frac{x\sin x}{x^2+5}=\lim_{x\to\infty}\frac{\cancel{x^2}\left(\frac{\sin x}{x}\right)}{\cancel{x^2}\left(1+\frac{5}{x^2}\right)}=\lim_{x\to\infty}\frac{\frac{\sin x}{x}}{1+\frac{5}{x^2}}=0.$$

2)
$$\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})}$$
$$=\lim_{h\to 0}\frac{\cancel{a}+h-\cancel{a}}{h(\sqrt{a+h}+\sqrt{a})}=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{a+h}+\sqrt{a})}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{2\sqrt{a}}.$$

This limit represents the derivative of the function $f(x)=\sqrt{x}$ at the point $a\neq 0$

3) Given $\varepsilon>0$, if $|x-0|<\delta$, then ~~since~~ $\sin^2 x\geq 0$

we have $\left|\frac{x}{1+\sin^2 x}\right|\leq |x|<\delta=\varepsilon$.

So $\forall\,\varepsilon>0$, $\exists\ \delta=\varepsilon>0$ such that

$\left|\frac{x}{1+\sin^2 x}-0\right|<\varepsilon$ whenever $|x-0|<\delta$,

which is equivalent to $\lim_{x\to 0}\frac{x}{1+\sin^2 x}=0$.

Exercise 2:-

1) Let us calculate the limit :-

$$\lim_{x\to 0} \frac{\tan^2 x + 2x}{x + x^2} = \lim_{x\to 0} \frac{\cancel{x}\left(\tan x \cdot \frac{\tan x}{x} + 2\right)}{\cancel{x}\,(1+x)}$$

$$= \lim_{x\to 0} \frac{\overset{\to 0}{\tan x} \cdot \overset{\to 1}{\frac{\tan x}{x}} + 2}{1 + \underset{\to 0}{x}} = 2.$$

Thus, for f to be continuous at 0, we should have $\lim_{x\to 0} \frac{\tan^2 x + 2x}{x + x^2} = 2 = c$

i.e. $c = 2$.

Thus, we infer that

$$f(x) = \begin{cases} \frac{\tan^2 x + 2x}{x + x^2}, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{-1, 0\} \\ 2, & x = 0 \end{cases}$$

is a continuous function at $x = 0$.

2) Set $f(x) = x\,2^x$, which is continuous on R and in particular on $[0,1]$.

Now, $f(0) = 0$, $f(1) = 1 \cdot 2^1 = 2$.

Thus $f(0) = 0 < 1 < 2 = f(1)$. So by the intermediate value theorem, there is a $c \in (0,1)$ such that $f(c) = c\,2^c = 1$, i.e. the equation $x\,2^x = 1$ admits a solution $c \in (0,1)$.

3) We have:

$$g'(x) = -e^{-x} \Rightarrow |g'(x)| = e^{-x} \leq 1, \forall x \geq 0$$

Thus $g(x) = e^{-x}$ has a bounded derivative on $[0, \infty)$. By the property we have proved in class, $g(x)$ is uniformly continuous in $[0, \infty)$.

Exercise 3:

1) $f(x) = x^4 - 2x^2, \Rightarrow D_f = \mathbb{R} = (-\infty, \infty)$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^4 - 2x^2 = \infty$$

$$\lim_{x \to \infty} x^4 - 2x^2 = \infty.$$

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

$f'(x) = 0 \Rightarrow x = 0$ or $x = 1$ or $x = -1$

those are the critical points of f.

$$f''(x) = 12x^2 - 4.$$

$f''(0) = -4 < 0 \Rightarrow f(0) = 0$ is a local max.

$f''(1) = 12 - 4 > 0 \Rightarrow f(1) = -1$ is a local min

$f''(-1) = 12 - 4 > 0 \Rightarrow f(-1) = -1$ is a local min.

x	$-\infty$		-1		0		1		∞
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$	∞	$\searrow$	-1	$\nearrow$	0	$\searrow$	-1	$\nearrow$	∞

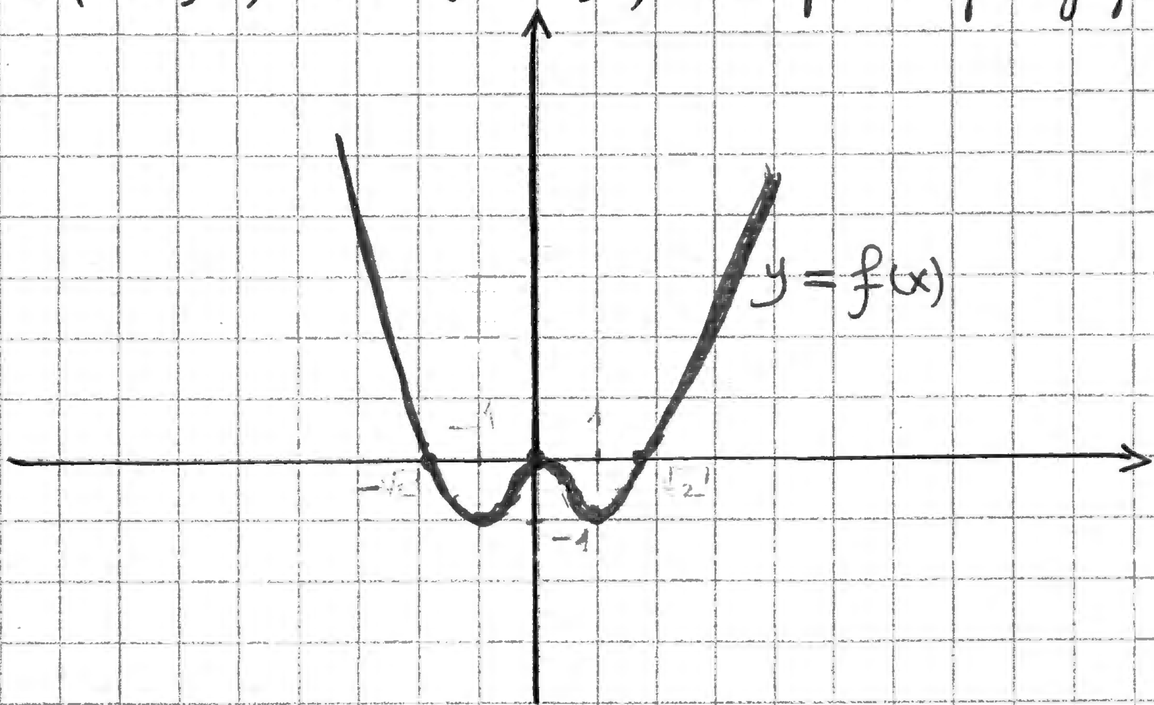
For the x and y intersepts, we have

$f(0) = 0 \longrightarrow (0,0)$ is a point of the graph.

$f(x) = 0 \Rightarrow x^2(x^2-2) = 0 \Rightarrow x = 0$ or

$x = \sqrt{2}$ or $x = -\sqrt{2}$.

$\Rightarrow (\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are points of the graph.



2) Consider the function

$$f(x) = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \dots + \frac{a_n}{n+1}x^{n+1}.$$

f is continuous in $[0,1]$ and differentiable on $(0,1)$, and

$f(0) = 0$, $f(1) = a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{n+1}$.

So, if $a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{n+1} = 0$, then $f(0) = f(1)$.

Thus Rolle's theorem is applicable and gives

$\exists\, c \in (0,1)$ such that $f'(c) = 0$.

Now

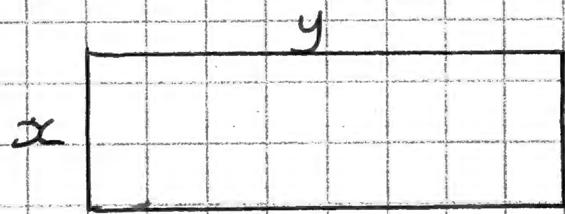
$$f'(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$\Rightarrow f'(c) = a_0 + c_1 c + a_2 c^2 + \dots + a_n c^n = 0$$

i.e. there is some $x \in [0,1]$ such that

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0.$$

3)



primeter $= 2x + 2y = 16$

Area : $A(x,y) = x \cdot y$

$2x + 2y = 16 \Rightarrow y = 8 - x$

$\Rightarrow A(x) = x(8-x) = -x^2 + 8x$

$A'(x) = -2x + 8 = 0 \Rightarrow x = 4$

$A''(x) = -2 < 0 \Rightarrow A(4) = 16$ is a local maximum.

Now, $x = 4$ and $y = 4$, so we get

$$x = y = 4.$$

This is a square with area $A = 16\ cm^2$.

So among rectangles of given primeter, the square is the the one with largest area.