الفصل الدراسي الثاني

١٤٢٧/٢٦هـ

King Saud University

Mathematics department-Women section

Final Examination, duration: 3 hours

282 Math

(a) Show that if $X = (x_n)$ is a bounded decreasing sequence, then

$$\lim (x_n) = \inf \{x_n : n \in \mathbb{N}\}$$

(b) If $\sup A \notin A$, show that $\sup A$ is a cluster point for A.

(a) Let $J_n = \left(0, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$, prove that $\bigcap_{n\in\mathbb{N}} J_n = \phi$.

(b) If $b > 1$, show that $\left(\frac{n}{b^n}\right)$ converges to zero.

(a) Using the continuity of $f(x) = e^x$, apply a sequential argument to prove that $\left(\sqrt[n]{e^{n+1}}\right)$ converges to e.

(b) If $f : [a, b] \to \mathbb{R}$ is continuous function. Show that there exists $x^* \in [a, b]$ such that

$$f(x^*) = \inf \{f(x) : a \leq x \leq b\}.$$

(c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for all $x \in Q$, show that $f(x) = 0$ for all $x \in \mathbb{R}$.

(a) Let $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 1 - x & x \notin \mathbb{Q}, \end{cases}$

1. Show that f is one to one , $f(f(x)) = x$, and find f^{-1}.
2. Find all the points at which f is continuous.

(b) Prove that if f is differentiable on $I = [a, b]$ and if k is a number between $f'(a)$ and $f'(b)$, then there is at least one point c in (a, b) such that $f'(c) = k$.

(c) Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if the derivative f' is never 0 on I, then either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

(a) Evaluate the following limit if it exists:

1. $\lim_{x\to 0^+} x^{\sin x}$
2. $\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x}, (x > 0)$.
3. $\lim_{x\to 0^+} \frac{\ln \sin x}{\ln \tan x}$

(b) Prove or disprove

1. If $K > 0$, $f : \mathbb{R} \to \mathbb{R}$ satisfies $\mid f(x) - f(y) \mid \leq K \mid x - y \mid$ for all $x, y \in \mathbb{R}$, then f is continuous on $\mathbb{R}$.
2. If $(x_n), (y_n)$ are both properly divergent, $y_n \neq 0$ for all $n \in N$, then $\left(\frac{x_n}{y_n}\right)$ is properly divergent.
3. Every function is a derivative of another function