Mathematical Models of Systems

- Development of mathematical models from schematics of physical systems.

by applying the fundamental physical laws of science and engineering.

- **Two Methods:**
  - *Transfer functions* in the frequency domain,
  - *State equations* in the time domain.

a. Block diagram representation of a system;

(a)

From mathematical model equations we will obtain the relationship between the system's output and input.

b. block diagram Representation of an interconnection of subsystems

(b)

Note: The input, *r(t)*, stands for *reference input*.
The output, *c(t)*, stands for *controlled variable*.

- *Transfer function* (mathematical function), is inside each block.

Modeling in Frequency Domain
Laplace Transform Review

- A system represented by a differential equation is difficult to model as a block diagram.
- A differential equation can describe the relationship between the input and output of a system.
- By using *Laplace transform* we can represent the input, output, and system as separate entities.

$$A(s)Y(s) = B(s)U(s)$$

Laplace transform can be defined as:

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Where $s = \sigma + j\omega$, a complex variable

Inverse Laplace transform:

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}ds = f(t)u(t)$$

Where, $u(t) = 1 \quad t > 0$

$\qquad\quad = 0 \quad t < 0$

Multiplication of *f(t)* by *u(t)* yields a time function that is zero for *t* < 0.

Laplace Transform Table

Table 2.1
Laplace transform table

Item no.	f(t)	F(s)
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Problem: Find the Laplace transform of

$$f(t) = Ae^{-at}u(t)$$

Solution:

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty Ae^{-at}e^{-st}dt$$

$$= A\int_0^\infty e^{-(s+a)t}dt = -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty}$$

$$= \frac{A}{s+a}$$

Laplace Transform Theorems

Table 2.2

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1]For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

[2]For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Inverse Laplace Transform

Problem: Find inverse Laplace Transform of

$$F_1(s) = \frac{1}{(s+3)^2}$$

Solution:

From item 3 and 5 of Table 2.1,

$$f_1(t) = e^{-3t}tu(t)$$

Frequency shift theorem item 4 of Table 2.2,

$$\mathscr{L}[e^{-at}f(t)] = F(s+a).$$

Inverse Laplace: Partial-Fraction Expansion

A partial-fraction expansion: transform of a complicated function to a sum of simpler terms for which we know the Laplace transform of each term.

Case 1. (Roots of the Denominator of F(s) Are Real and Distinct)

Problem: $$F_1(s) = \frac{s^3 + 4s^2 + 6s + 5}{s^2 + 3s + 2} = \frac{N(s)}{D(s)}$$

The order of N(s) > order of D(s) we must perform the division until we obtain a remainder whose numerator is of order less than its denominator

Solution:

$$\begin{array}{r|l} & s+1 \\ \hline s^2+3s+2 & s^3+4s^2+6s+5 \\ & -(s^3+3s^2+2s) \\ \hline & s^2+4s+5 \\ & -(s^2+3s+2) \\ \hline & s+3 \end{array}$$

$$F_1(s) = s + 1 + \frac{s+3}{s^2+3s+2}$$

Taking the inverse Laplace transform, (Table 2.2 Item 7)

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathfrak{I}^{-1}\left[\frac{s+3}{(s+1)(s+2)}\right]$$

Partial-Fraction Expansion

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

To find K_1, multiply by $(s+1)$. Thus

$$\frac{s+3}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)}$$

Letting $s = -1$, $K_1 = 2$

Similarly, $K_2 = -2$

$$f(t) = (2e^{-t} - 2e^{-2t})u(t)$$

Final solution: $$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + (2e^{-t} - 2e^{-2t})u(t)$$

Laplace Transform Solution of a Differential Equation

Problem: Solve for *y(t)*, if all initial conditions are zero.

$$\frac{d^2y}{dt^2}+12\frac{dy}{dt}+32y=32u(t)$$

Solution: The Laplace transform is,

(Table 2.2 Item 8)

$$s^2Y(s)+12sY(s)+32Y(s)=\frac{32}{s}$$

$$Y(s)=\frac{32}{s(s^2+12s+32)}=\frac{32}{s(s+4)(s+8)}$$

$$=\frac{K_1}{s}+\frac{K_2}{(s+4)}+\frac{K_3}{(s+8)}$$

$$K_1=\left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0}=1$$

$$K_2=\left.\frac{32}{s(s+8)}\right|_{s\to -4}=-2$$

$$K_3=\left.\frac{32}{s(s+4)}\right|_{s\to -8}=1$$

Hence, $Y(s)=\frac{1}{s}-\frac{2}{(s+4)}+\frac{1}{(s+8)}$

inverse Laplace transform

Taking inverse Laplace transform, we get

$$y(t)=(1-2e^{-4t}+e^{-8t})u(t)$$

Inverse Laplace: Partial-Fraction Expansion

Case 2. (Roots of the Denominator of F(s) Are Real and Repeated)

Problem: Find inverse Laplace transform of

$$F(s) = \frac{2}{(s+1)(s+2)^2} \quad (1)$$

Solution:

We can write the partial-fraction expansion as a sum of terms

reduced multiplicity

$$F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)} \quad (2)$$

$$K_1 = \left.\frac{2}{(s+2)^2}\right|_{s\to -1} = 2$$

To find K_2, multiply (1) = (2) by $(s+2)^2$

$$\frac{2}{(s+1)} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3 \quad (3)$$

Letting $s \to -2, we\ obtain\ K_2 = -2$

To find K_3, differentiate (3) w.r.t. s:

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2}K_1 + K_3$$

Letting $s \to -2, we\ obtain\ K_3 = -2$

Therefore, inverse Laplace transform is:

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

For repeated roots with multiplicity r, we have

$$F(s) = \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \cdots + \frac{K_r}{(s+p_1)}$$

$$K_i = \left.\frac{1}{(i-1)!}\frac{d^{i-1}F_1(s)}{ds^{i-1}}\right|_{s\to -p_1} \qquad i = 1, 2, \ldots, r;$$

$$F_1(s) = (s+p_1)^r F(s)$$

Inverse Laplace: Partial-Fraction Expansion

Case 3. (Roots of the Denominator of F(s) Are Complex or Imaginary)

Problem: Find inverse Laplace transform of $F(s) = \dfrac{3}{s(s^2+2s+5)}$

Solution:

This function can be expanded in the following form:

$$\frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5} \quad (1)$$

K_1 is found in the usual way: $\left.\dfrac{3}{s^2+2s+5}\right|_{s\to 0} = \dfrac{3}{5} \equiv K_1$

To find K_2 and K_3 :

Multiply (1) by $s(s^2+2\,s+5)$, and put $K_1 = {}^3/_5$

$$\Rightarrow \quad 3 = \frac{3}{5}(s^2+2\,s+5) + K_2\,s^2 + K_3\,s$$

$$\Rightarrow \quad 3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3$$

Balancing coefficients(matching)

$$K_2 + \frac{3}{5} = 0$$

$$K_3 + \frac{6}{5} = 0$$

$$\Rightarrow \quad K_2 = -\frac{3}{5}, \text{ and } K_3 = -\frac{6}{5}$$

Hence, $F(s) = \dfrac{3/5}{s} - \dfrac{3}{5}\dfrac{s+2}{s^2+2s+5}$

Using Item 7 in Table 2.1 and Items 2 and 4 in Table 2.2, we get

$$\Im[Ae^{-at}\cos\omega t] = \frac{A(s+a)}{(s+a)^2+\omega^2}$$

$$\Im[Be^{-at}\sin\omega t] = \frac{B\omega}{(s+a)^2+\omega^2}$$

Adding, $\Im[Ae^{-at}\cos\omega t + Be^{-at}\sin\omega t] = \dfrac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$

We have, $s^2 + 2s + 5 = s^2 + 2s + 1 + 4 = (s+1)^2 + 2^2$

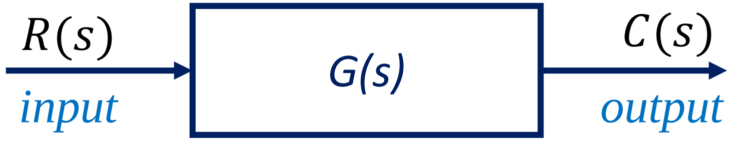
$$\Rightarrow \quad a = 1 \text{ and } \omega = 2$$

$$\Rightarrow \quad F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{(s+1)+(1/2)(2)}{(s+1)^2+2^2}$$

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Transfer Function

- A *Transfer Function* is the ratio of the output of a system to the input of a system. It allows us to algebraically combine mathematical representations of subsystems to yield a total system representation.



- General n[th] order, linear time-invariant differential equation:

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \ldots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1} r(t)}{dt^{m-1}} + \ldots + b_0 r(t)$$

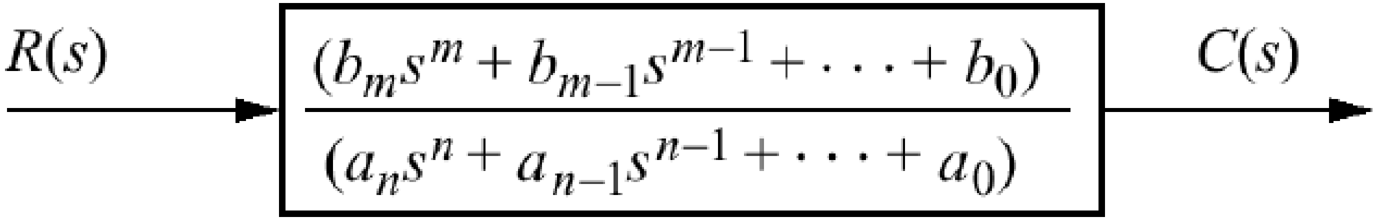
c: output, r: input

Taking Laplace transform,

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \ldots + a_0 C(s) + [\text{Initial condition is zero}]$$
$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \ldots + b_0 R(s) + [\text{Initial condition is zero}]$$

Transfer function:

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0)}$$



Block diagram of a transfer function

We can find the output

$$C(s) = G(s)R(s)$$

Transfer Function for a Differential Equation

Problem 1: Find the transfer function represented by $$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Solution:

Taking Laplace transform and assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s)$$

Transfer function, G(s), $$\Rightarrow G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Problem 2: Find the transfer function represented by $$\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r.$$

Solution:

$$s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$$

$$\Rightarrow G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

Problem Solving

Problem: Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s+4)(s+8)}$$

Solution:

The input (ramp)

$$C(s) = R(s)G(s) = \frac{1}{s^2} \times \frac{s}{(s+4)(s+8)} = \frac{1}{s(s+4)(s+8)}$$

$$\Rightarrow C(s) = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+8)}$$

$$A = \left.\frac{1}{(s+4)(s+8)}\right|_{s \to 0} = \frac{1}{32}$$

$$B = \left.\frac{1}{s(s+8)}\right|_{s \to -4} = -\frac{1}{16}$$

$$C = \left.\frac{1}{s(s+4)}\right|_{s \to -8} = \frac{1}{32}$$

Hence,

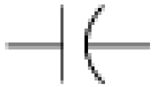
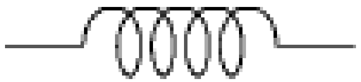
$$c(t) = \frac{1}{32} - \frac{1}{16} e^{-4t} + \frac{1}{32} e^{-8t}$$

Electric Network Transfer Functions

Apply the transfer function to the mathematical modeling of electronic circuits including passive networks and O-Amp circuits.

Table 2.3

Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

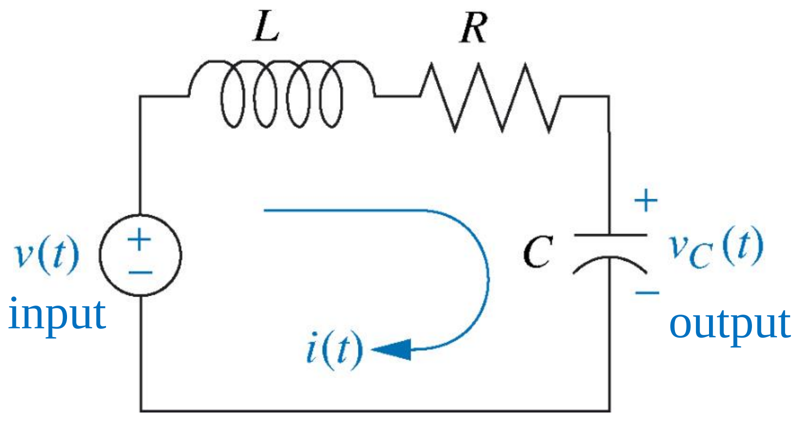
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

Transfer Function: Single Loop

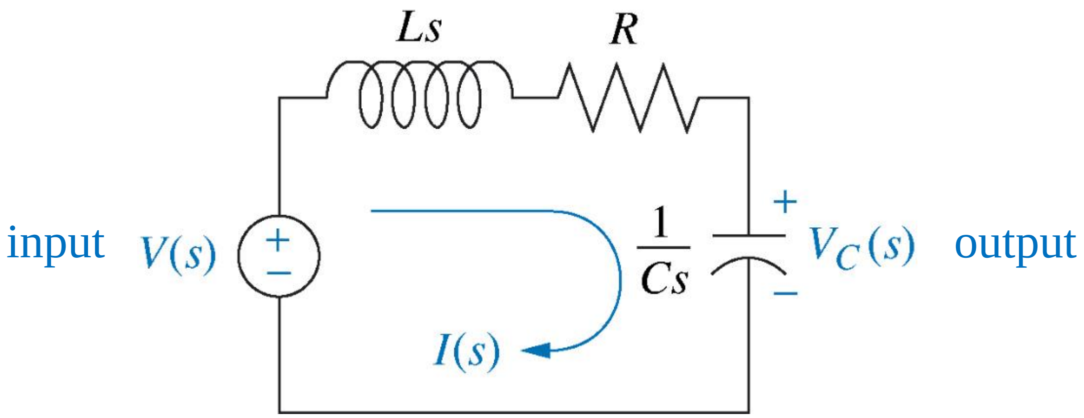
1

Problem: Find the transfer function relating capacitor voltage, $V_c(s)$, to input voltage, $V(s)$.



RLC network

Laplace-transform



Laplace-transformed network

Summing the voltages around the loop,

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = V(t)$$

take the Laplace transform

$$V_C(s) = \frac{1}{C\,s} I(s) \qquad V_L(s) = L\,s\,I(s) \qquad V_R(s) = R\,I(s)$$

Capacitor Inductor Resistor

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s) \Rightarrow \frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

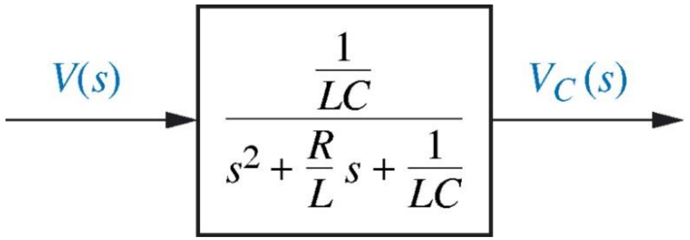
We know, $V_c(s) = I(s)\frac{1}{Cs} \Rightarrow \frac{V_c(s)}{V(s)} = \frac{I(s)}{V(s)} \times \frac{1}{Cs}$

$$\Rightarrow V_c(s) = \frac{V(s)}{Ls + R + \frac{1}{Cs}} \times \frac{1}{Cs}$$

$$\Rightarrow \frac{V_c(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}} \times \frac{1}{Cs} = \frac{Cs}{s^2LC + sRC + 1} \times \frac{1}{Cs}$$

$$= \frac{1}{s^2LC + sRC + 1}$$

$$= \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$



Transfer Function: Single Node (2)

Transfer functions also can be obtained using Kirchhoff's current law and summing currents flowing from nodes. currents leaving the node are positive and currents entering the node are negative.

$$\sum I_{in} = \sum I_{out} \;\Rightarrow\; I_c(s) = I_{RL}(s)$$

Same current

$$\Rightarrow I_c(s) - I_{RL}(s) = 0 \;\Rightarrow\; \frac{V_c}{Z_c} - \frac{V_{RL}}{Z_{RL}} = 0$$

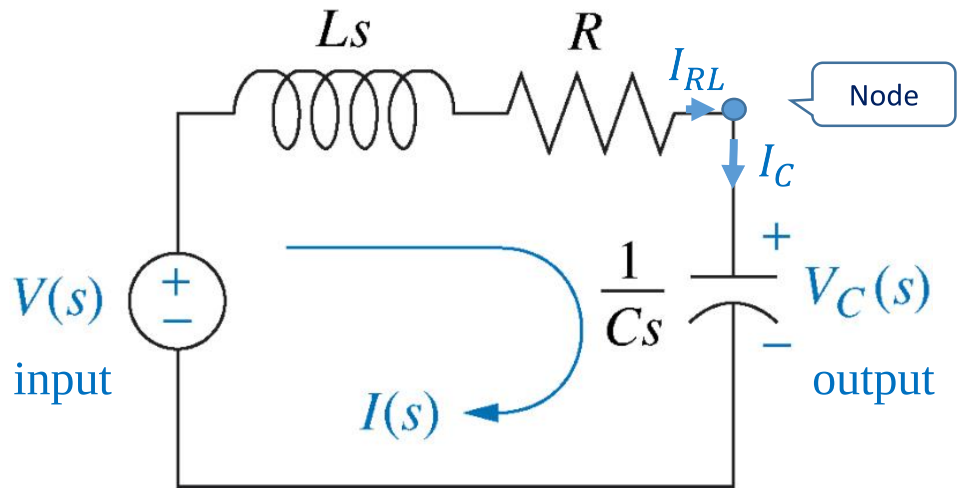
$V_{RL} = -(V_c - V)$

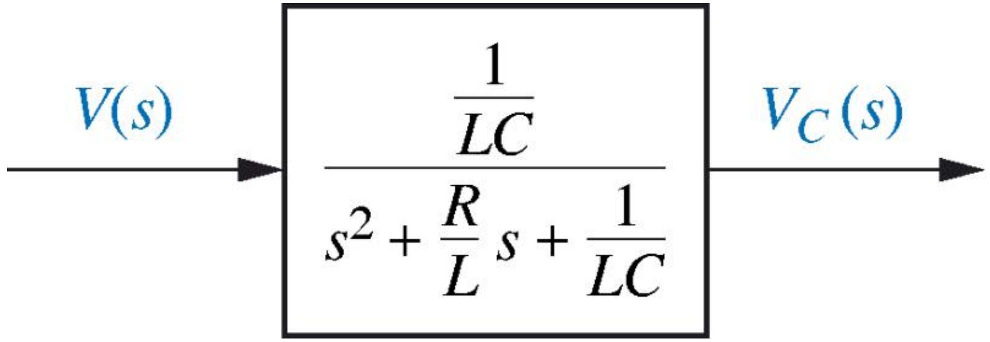
$$\Rightarrow \frac{V_c(s)}{\frac{1}{Cs}} + \frac{V_c(s) - V(s)}{R + Ls} = 0$$

$$\Rightarrow V_c(s)\left(Cs + \frac{1}{R + Ls}\right) = \frac{V(s)}{R + Ls}$$

$$\Rightarrow \frac{V_c(s)}{V(s)} = \frac{1}{s^2LC + sRC + 1}$$

$$= \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

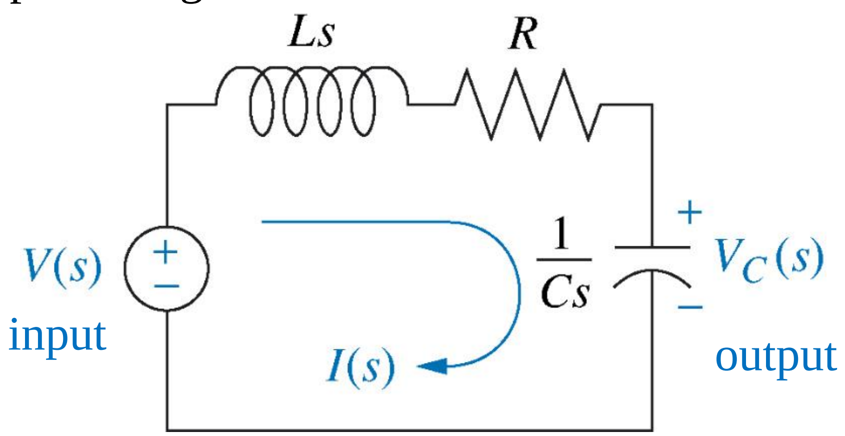




Transfer Function: Single Loop via Voltage Division

3

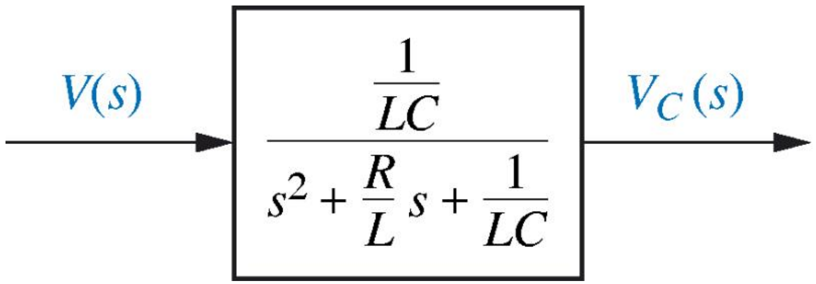
Voltage across capacitor is some proportion of the input voltage.

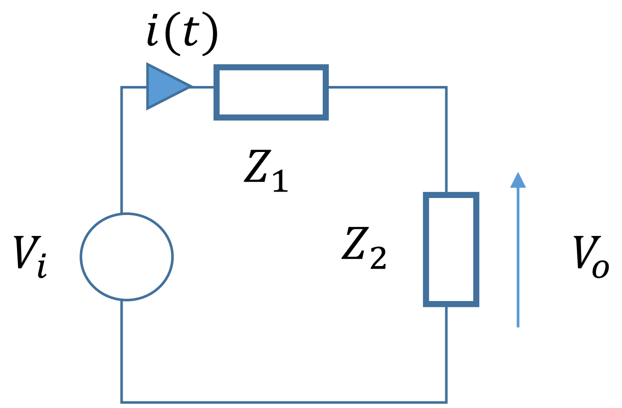


→ $\frac{\text{Impedance of the capacitor}}{\text{Sum of the impedances}}$

$$V_c(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s)$$

$$\Rightarrow \frac{V_c(s)}{V(s)} = \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$





$$V_i = (Z_1 + Z_2)\, i(t) \quad (1)$$

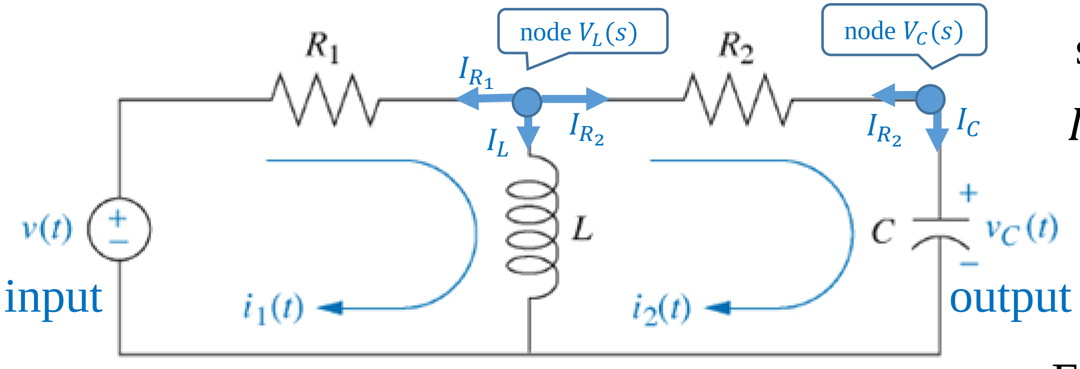
$$V_o = Z_2\, i(t) \quad (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{V_o}{V_i} = \frac{Z_2}{(Z_1 + Z_2)}$$

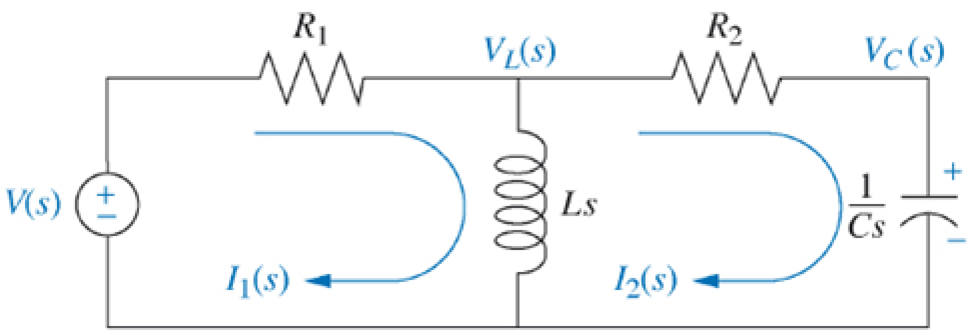
Which one is the easiest? Method 1, method 2, or method 3?

Complex Circuits via Nodal Analysis$_1$

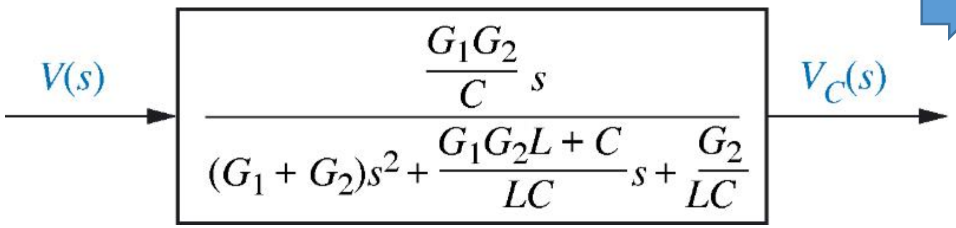
Complex electrical networks (those with multiple loops and nodes). We use nodal analysis to find the *transfer function* $\frac{V_C(s)}{V(s)}$



(a)



(b)



sum currents at the nodes

Current from node $V_L(s)$

$$I_{R_1} + I_L + I_{R_2} = 0 \;\Rightarrow\; \frac{V_L(s)-V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s)-V_C(s)}{R_2} = 0$$

And $I_C + I_{R_2} = 0 \;\Rightarrow\; CsV_C(s) + \frac{V_C(s)-V_L(s)}{R_2} = 0$

Current from node $V_C(s)$

Expressing resistances as conductances, G=1/R

$$\left(G_1 + G_2 + \frac{1}{Ls}\right)V_L(s) - G_2V_C(s) = V(s)G_1 \qquad (1)$$

$$-G_2V_L(s) + (G_2 + Cs)V_C(s) = 0 \;\Rightarrow\; V_L(s) = \frac{G_2 + Cs}{G_2} \qquad (2)$$

(2) in (1) $\Rightarrow$

$$\frac{V_C(s)}{V(s)} = \frac{G_1G_2Ls}{(G_1+G_2)LCs^2 + \left(C + G_2(G_1+G_2)L - {G_2}^2L\right)S + G_2}$$

Divide by LC

$\Rightarrow$ Transfer function:

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1G_2}{C}s}{(G_1+G_2)s^2 + \frac{G_1G_2L+C}{LC}s + \frac{G_2}{LC}}$$

Complex Circuits - Mesh Equations via Inspection$_2$

For Mesh 1: $\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around} \\ \text{Mesh 1} \end{bmatrix} \times I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh2} \end{bmatrix} \times I_2(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh3} \end{bmatrix} \times I_3(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{bmatrix}$

For Mesh 2: $-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh2} \end{bmatrix} \times I_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around} \\ \text{Mesh 2} \end{bmatrix} \times I_2(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 2 and mesh3} \end{bmatrix} \times I_3(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{bmatrix}$

For Mesh 3: $-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh3} \end{bmatrix} \times I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 2 and mesh3} \end{bmatrix} \times I_2(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around} \\ \text{Mesh 3} \end{bmatrix} \times I_3(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 3} \end{bmatrix}$

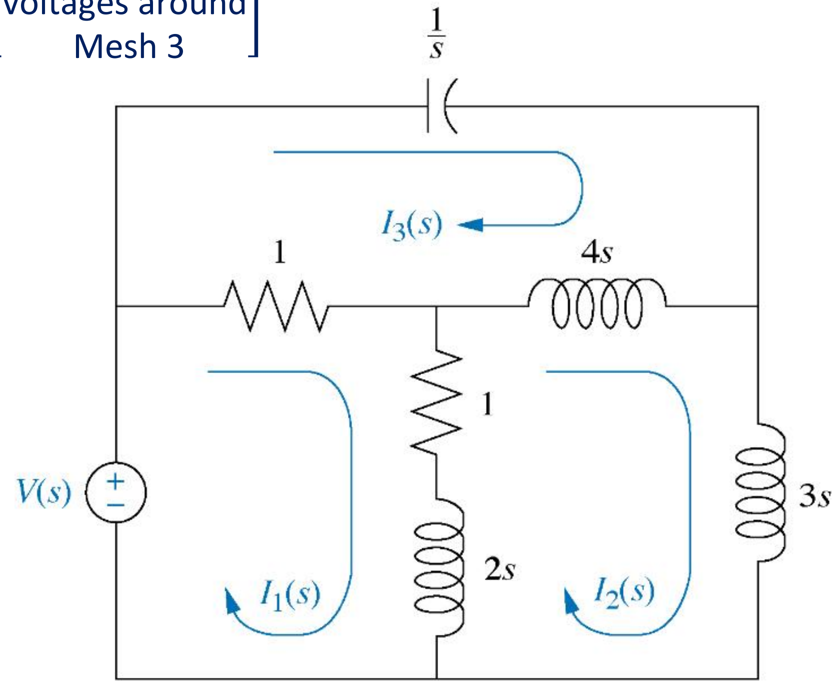
Similarly, Meshes 3, we obtain

For Mesh 1: $+(2s+2)I_1(s)-(2s+1)I_2(s)-I_3(s)=V(s)$

For Mesh 2: $-(2s+1)I_1(s)+(9s+1)I_2(s)-4sI_3(s)=0$

For Mesh 3: $-I_1(s)-4sI_2(s)+\left(4s+1+\frac{1}{s}\right)I_3(s)=0$

which can be solved simultaneously for any desired transfer function, for example, $I_3(s)/V(s)$



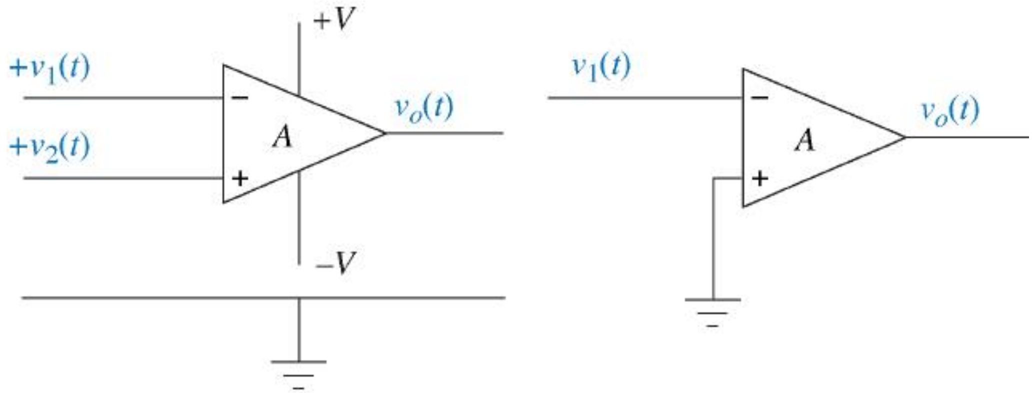
Three loop electrical network

Operational Amplifier

An *operational amplifier* is an electronic amplifier used as a basic building block to implement transfer functions. It has the following characteristics:

1. Differential input, $v_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty\ (ideal)$
3. Low output impedance, $Z_0 = 0\ (ideal)$
4. High constant gain amplification, $A = \infty\ (ideal)$

The output, $v_0(t)$, is given by: $v_0(t) = A(v_2(t) - v_1(t))$

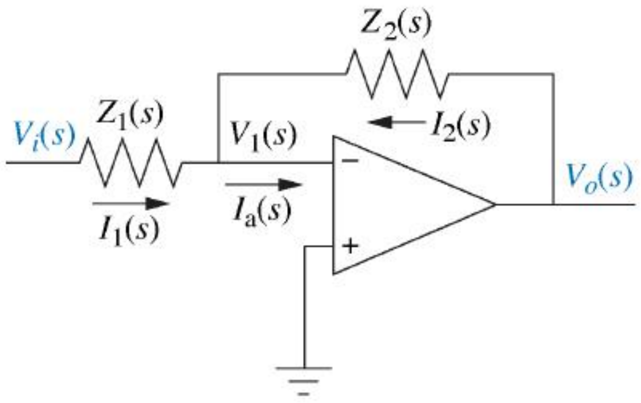


(a) (b)

a. Operational amplifier;

b. schematic for an inverting operational amplifier;

c. Inverting operational amplifier configured for transfer function realization. Typically, the amplifier gain, A, is omitted.



(c)

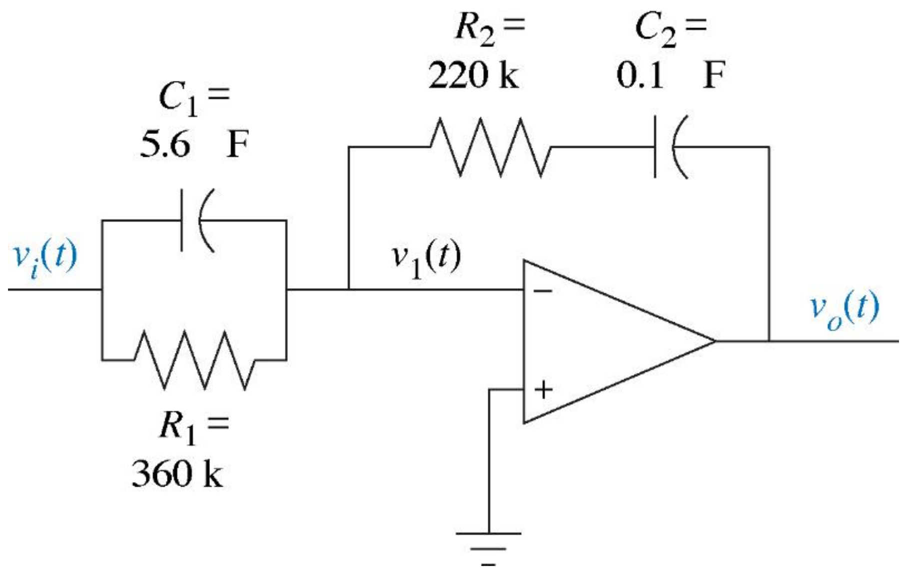
$I_1(s) = -I_2(s)$, as $I_a(s) = 0$, because of high input impedance

$$V_i(s) = Z_1(s) I_1(s)$$
$$V_o(s) = Z_2(s) I_2(s)$$
$$I_1(s) = -I_2(s)$$

$$\Rightarrow \quad \frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

Problem Solving
Inverting Operational Amplifier

Problem: Find the transfer function, $\frac{V_0(s)}{V_i(s)}$, for the circuit below.



Solution:

For parallel components, $Z_1(s)$ is the reciprocal of the sum of the admittances.

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6\times10^{-6} s + \frac{1}{360\times10^{3}}}$$

For serial components, $Z_2(s)$ is the sum of the impedances.

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220\times10^{3} + \frac{10^{7}}{s}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -1.232\times\frac{s^2 + 45.95s + 22.55}{s}$$

Non-inverting Operational Amplifier

Using voltage division,

$$V_0(s) = [Z_1(s) + Z_2(s)]\, I(s)$$
$$V_1(s) = Z_1(s)\, I(s)$$

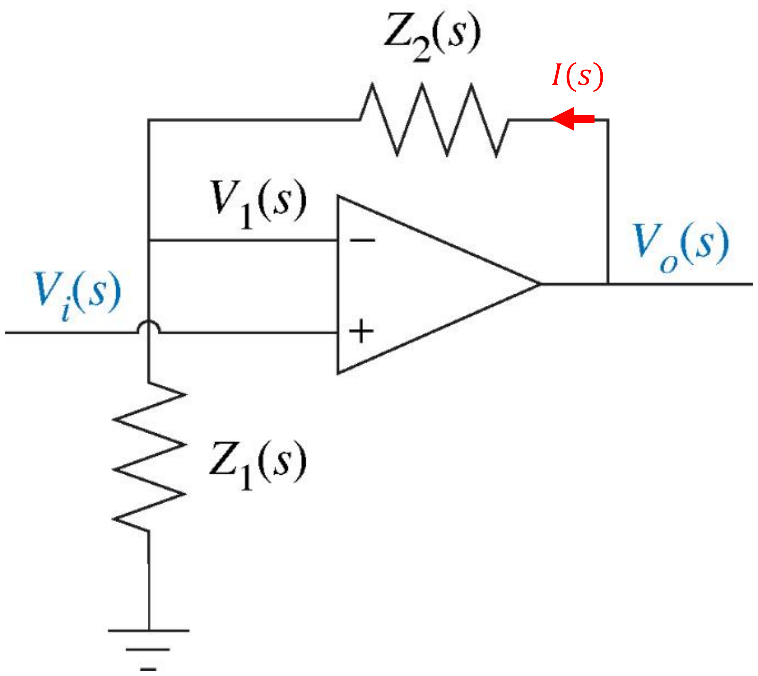
$$\Rightarrow V_1(s) = \frac{Z_1(s)}{Z_1(s)+Z_2(s)} V_o(s)$$

We have: $V_o(s) = A(V_i(s) - V_1(s))$ $\Rightarrow$ $V_0(s) = A\left[V_i(s) - \frac{Z_1(s)}{Z_1(s)+Z_2(s)} V_o(s)\right]$

$$\Rightarrow V_0(s)\left[1 + A\frac{Z_1(s)}{Z_1(s)+Z_2(s)}\right] = A\, V_i(s)$$

$$\Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s)+Z_2(s))}$$

For large A, we disregard '1' in the denominator. $\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s)+Z_2(s)}{Z_1(s)}$



Problem Solving
Non-Inverting Operational Amplifier

PROBLEM: Find the transfer function, $V_0(s)/Vi(s)$, for the Non-inverting operational amplifier circuit

SOLUTION:

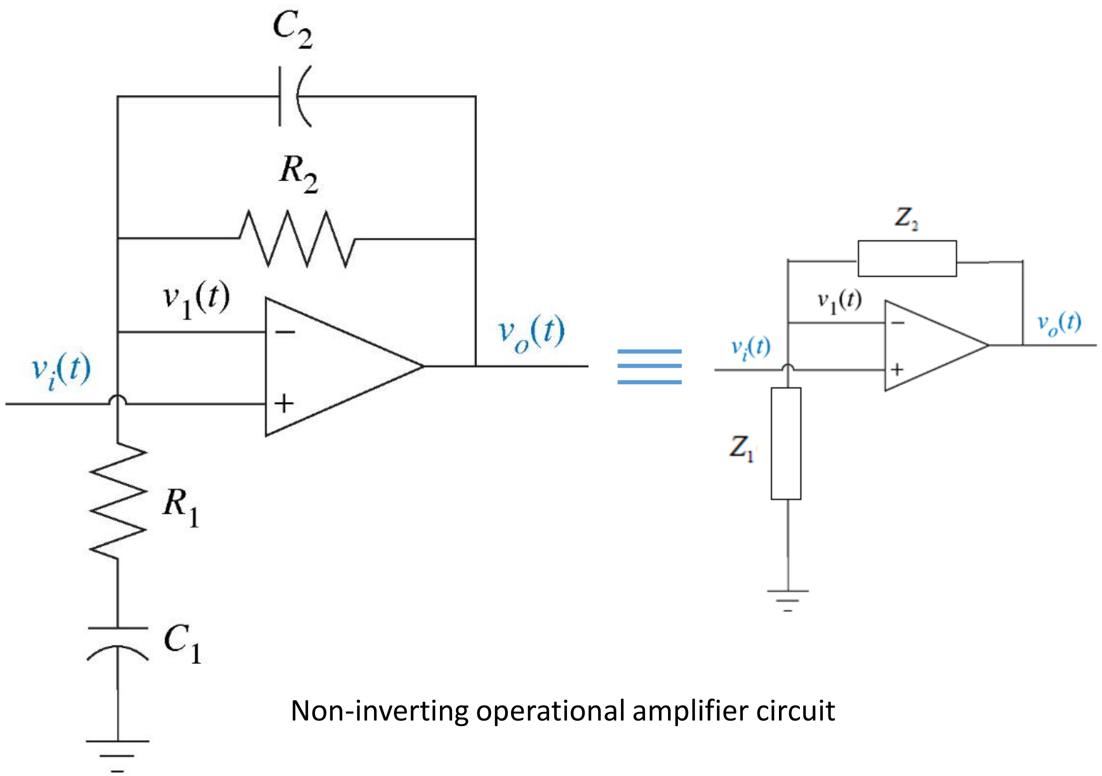
We find each of the impedance functions,

$$Z_1(s) = R_1 + \frac{1}{C_1 s} = \frac{R_1 C_1 s + 1}{C_1 s} \quad \text{and} \quad Z_2(s) = \frac{1}{C_2 s + \frac{1}{R_2}} = \frac{R_2}{R_2 C_2 s + 1}$$

Now use the following equation: $\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$

$$\frac{V_0(s)}{V_1(s)} = \left[\frac{R_1 C_1 s + 1}{C_1 s} + \frac{R_2}{R_2 C_2 s + 1}\right] \frac{C_1 s}{R_1 C_1 s + 1} = 1 + \frac{R_2 C_1 s}{(R_2 C_2 s + 1)(R_1 C_1 s + 1)}$$

$$\Rightarrow \quad \frac{V_0(s)}{V_1(s)} = 1 + \frac{R_2 C_1 s}{R_1 R_2 C_1 C_2 s^2 + R_2 C_2 s + R_1 C_1 s + 1}$$



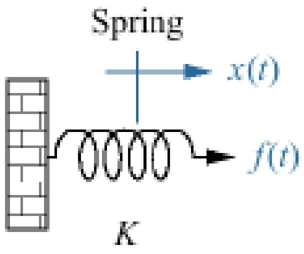
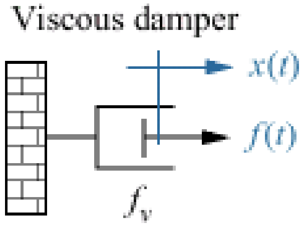
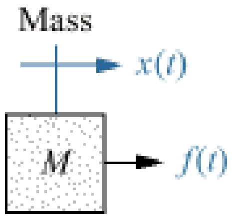
Non-inverting operational amplifier circuit

Substituting yields $$\frac{V_o(s)}{V_i(s)} = \frac{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_2 + C_1 R_1)s + 1}{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_1)s + 1}$$

Translational Mechanical System Transfer Functions

Mechanical systems (like electrical networks) have three passive linear components: **Spring** and the **mass** (energy-storage elements); and **viscous damper** (dissipates energy).

Table 2.4
Force-velocity, force-displacement, and impedance translational relationships
for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

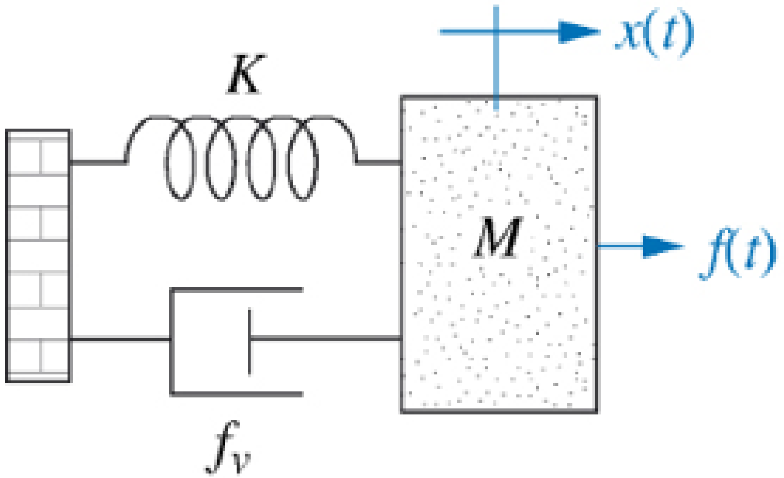
K: Spring constant

f_V: Coefficient of viscous friction

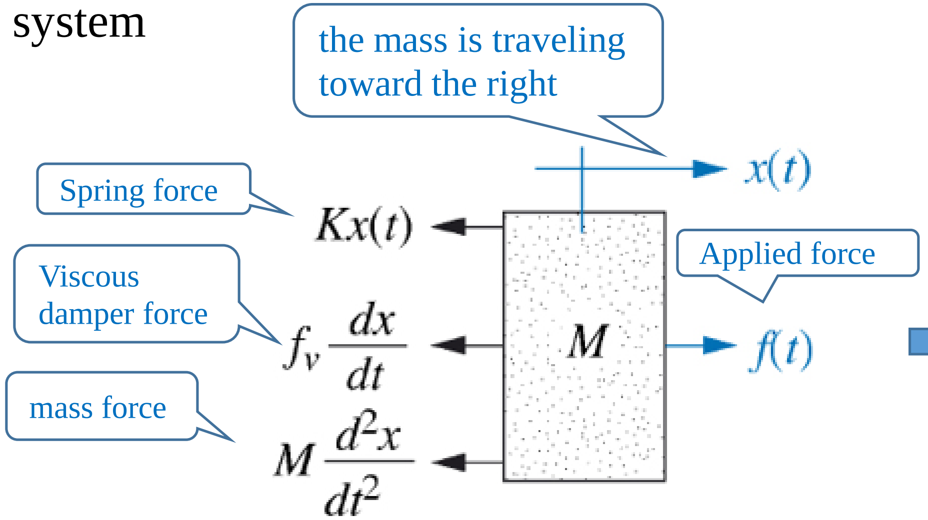
M: Coefficient of mass

Transfer Functions: One Degree of Freedom

Find the transfer function, *X(s)/F(s)*, for the system

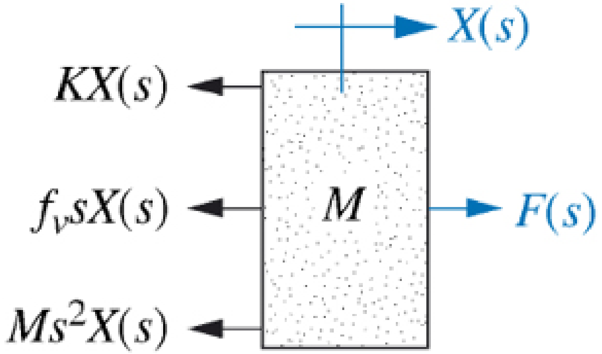


Mass, spring, and damper system



Free-body diagram of mass, spring, and damper system;

LT



Transformed free body diagram

- All the forces impede (obstruct and block) the motion and act to oppose the applied force.

Differential equation of motion (Newton's law)

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

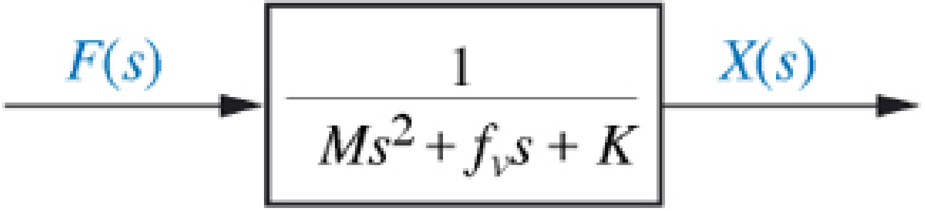
LT

Sum of impedances × X(s) = Sum of applied forces (zero initial conditions)

$$(K + f_v s + Ms^2) \times X(s) = F(s)$$

Solving for the transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

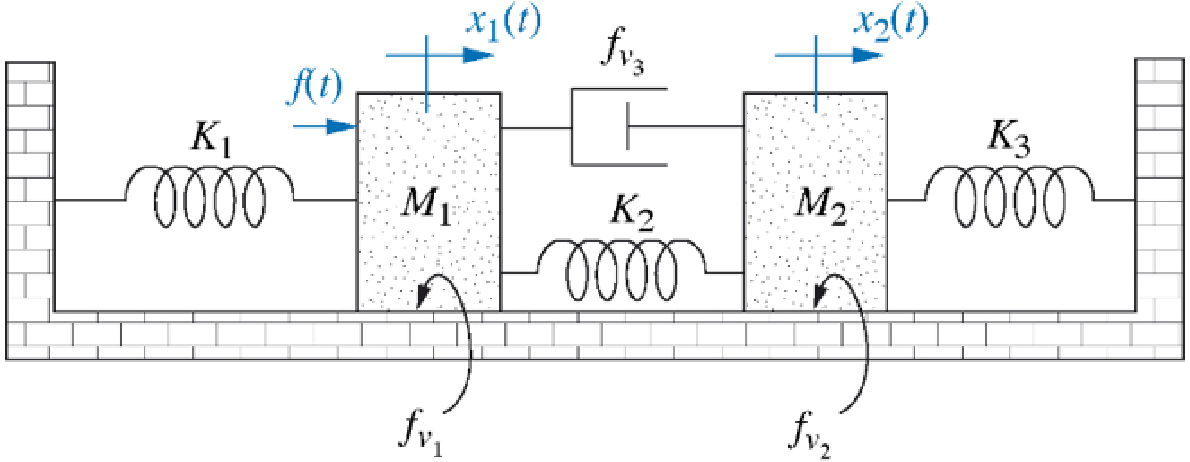


Transfer function

Transfer Functions: Two Degrees of Freedom

Number of differential equations required to describe the system is equal to the number of *linearly independent* motions (*degrees of freedom*).

- Find the transfer function, $X_2(s)/F(s)$, for the system

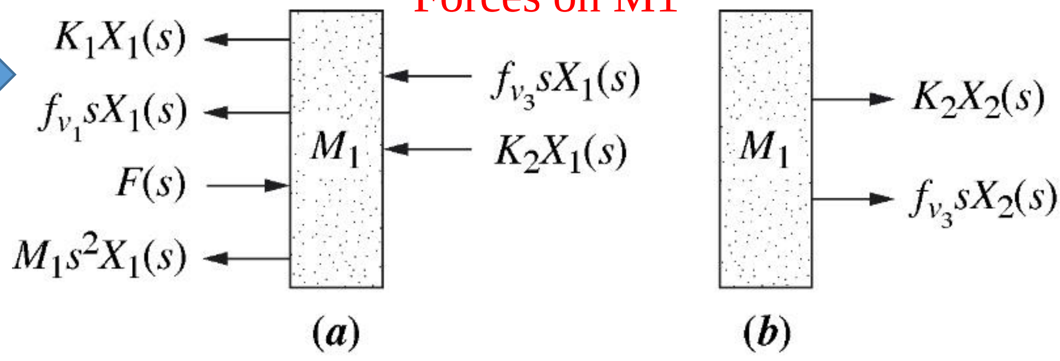


Two-degrees-of-freedom translational mechanical system

Two-degrees-of-freedom : since Each mass can be moved in the horizontal direction while the other is held still.

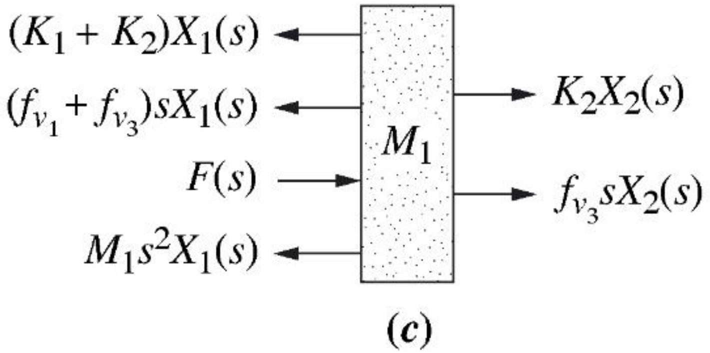
a. Forces on M1 due only to motion of M1
b. forces on M1 due only to motion of M2
c. all forces on M1

Forces on M1



hold M2 and move M1

hold M1 and move M2



total force on *M1* *(superposition or sum)*

The Laplace transform of the equation of motion of M1

$$[M_1s^2+(f_{v1}+f_{v3})s+(K_1+K_2)]X_1(s)-(f_{v3}s+K_2)X_2(s)=F(s) \quad (1)$$

$$A\,X_1(s)-BX_2(s)=F \quad (1)$$

Transfer Functions: Two Degrees of Freedom Continued

The Laplace transform of the equation of motion of M2

$$-(f_{v3}s+K_2)X_1(s)+[M_2s^2+(f_{v2}+f_{v3})s+(K_2+K_3)]X_2(s)=0$$

$$-C\,X_1(s)+DX_2(s)=0 \qquad X_1(s)=\frac{D}{C}X_2(s) \quad (2)$$

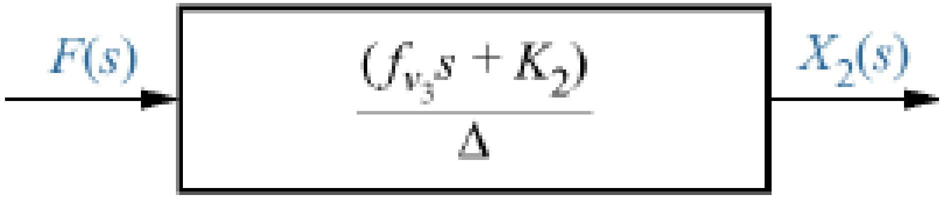
(2) in (1) ⇒ $A\frac{D}{C}X_2(s)-BX_2(s)=F$ ⇒ $\frac{X_2(s)}{F(s)}=\frac{C}{AD-CB}$

Transfer function: $\frac{X_2(s)}{F(s)}=G(s)=\frac{(f_{v_3}s+K_2)}{\Delta}$

Determinant

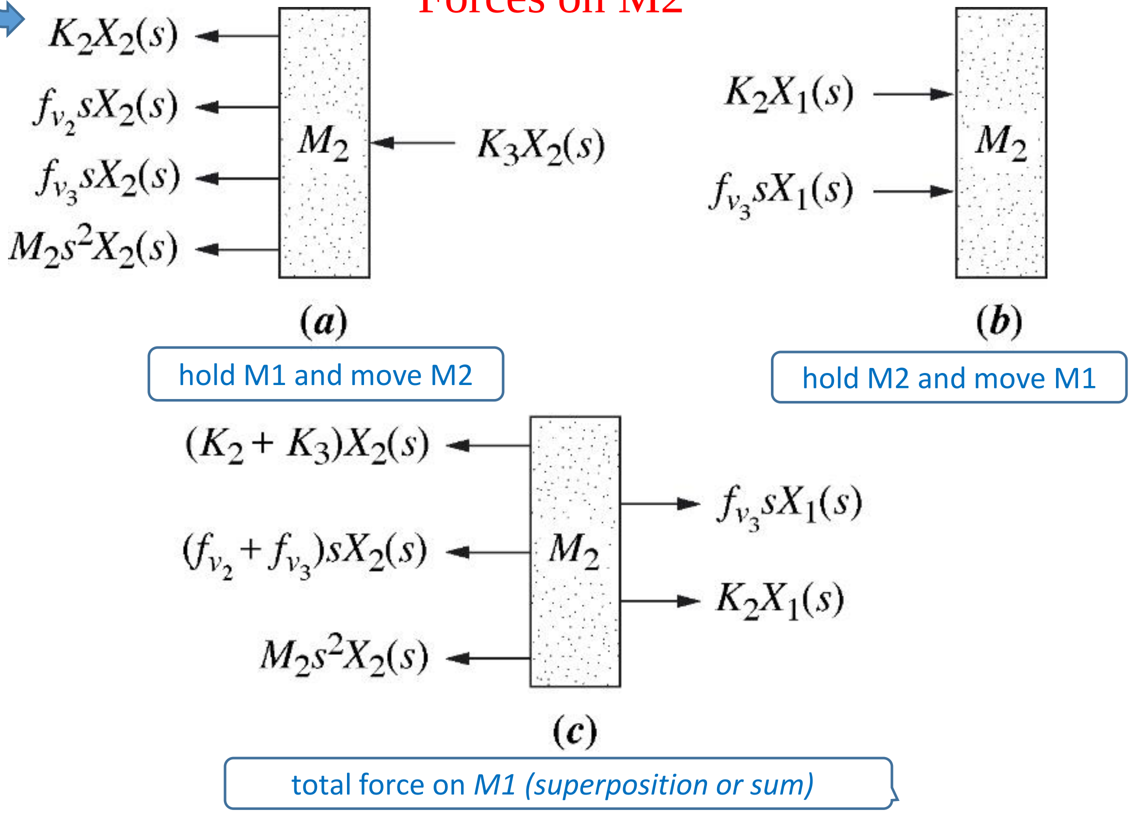
where

$$\Delta=\begin{vmatrix}[M_1s^2+(f_{v_1}+f_{v_3})s+(K_1+K_2)] & -(f_{v_3}s+K_2)\\ -(f_{v_3}s+K_2) & [M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)]\end{vmatrix}$$



a. Forces on M2 due only to motion of M2;
b. forces on M2 due only to motion of M1;
c. all forces on M2

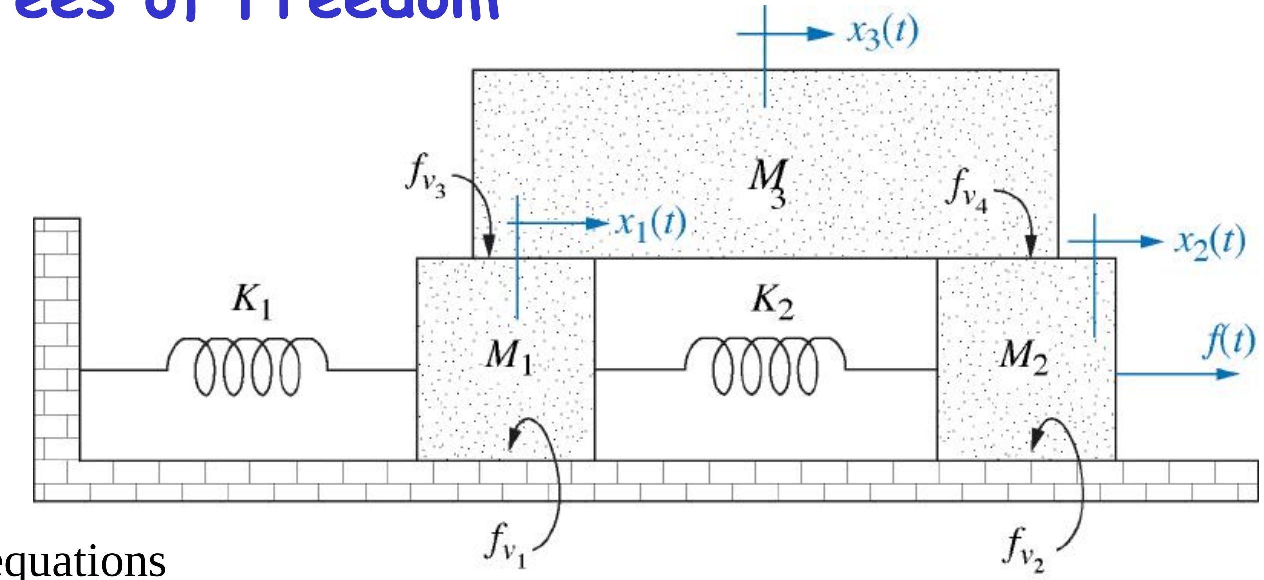
Forces on M2



Transfer Functions:
Three Degrees of Freedom

- Write, the equations of motion for the mechanical network

- The system has three degrees of freedom, since each of the three masses can be moved independently while the others are held still.



- The form of the equations will be similar to electrical mesh equations

For M1:

$$\begin{bmatrix}\text{Sum of impedances connected to the motion at } x_1\end{bmatrix} X_1(s) - \begin{bmatrix}\text{Sum of impedances between } x_1 \text{ and } x_2\end{bmatrix} X_2(s) - \begin{bmatrix}\text{Sum of impedances between } x_1 \text{ and } x_3\end{bmatrix} X_3(s) = \begin{bmatrix}\text{Sum of applied forces at } x_1\end{bmatrix}$$

Similarly, for M2 and M3, we obtain

M_1: $$[M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - K_2 X_2(s) - f_{v3} s X_3(s) = 0$$

M_2: $$-K_2 X_1(s) + [M_2 s^2 + (f_{v2} + f_{v4})s + K_2]X_2(s) - f_{v4} s X_3(s) = F(s)$$

M_3: $$-f_{v3} s X_1(s) - f_{v4} s X_2(s) + [M_3 s^2 + (f_{v3} + f_{v4})s]X_3(s) = 0$$

Nonlinearity

Linear systems have two properties: (1) *additivity*, and (2) *homogeneity*.

1. *Additivity* (*superposition*):

   If $r_1(t) \rightarrow c_1(t)$ and $r_2(t) \rightarrow c_2(t)$, then $r_1(t) + r_2(t) \rightarrow c_1(t)+c_2(t)$

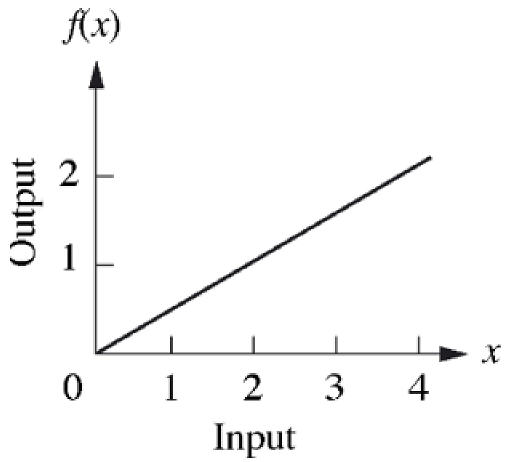
2. *Homogeneity*:

   If $r_1(t) \rightarrow c_1(t)$, then $Ar_1(t) \rightarrow Ac_1(t)$

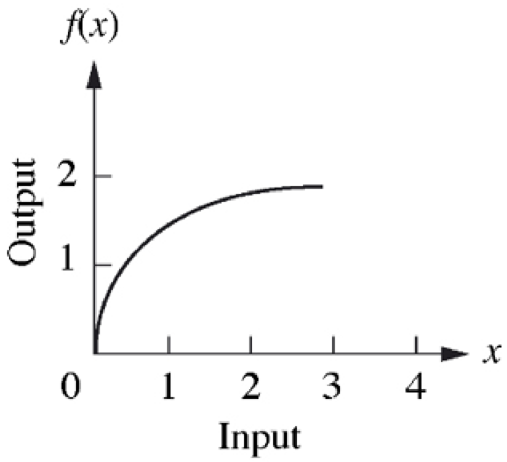
r(t): Input

c(t): Output

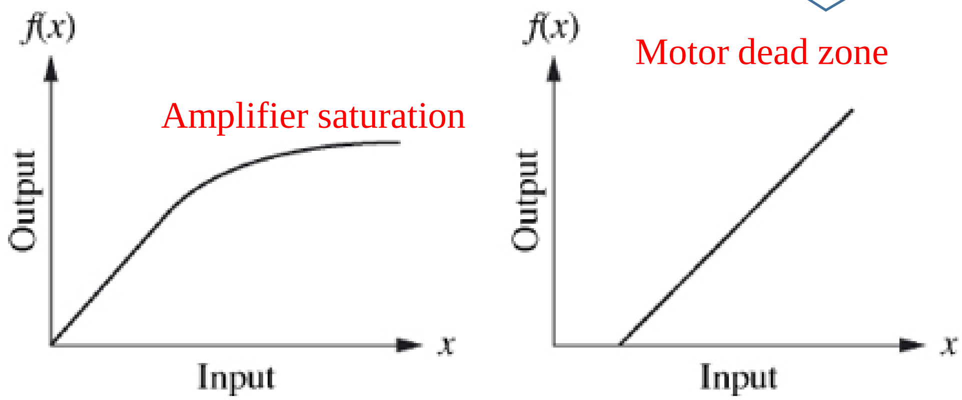
motor does not respond at very low input voltages due to frictional forces exhibits a nonlinearity called *dead zone*



Linear system



Nonlinear system



Some physical nonlinearities

Linearization

- If the system is nonlinear, we must linearize the system before we can find the transfer function.
- Making linear approximation to a nonlinear system.
- For a nonlinear system operating at point A: $[x_0 \, , f(x_0)]$

Linear approximation :

related by the Slope m_A (line) of the curve at the point A

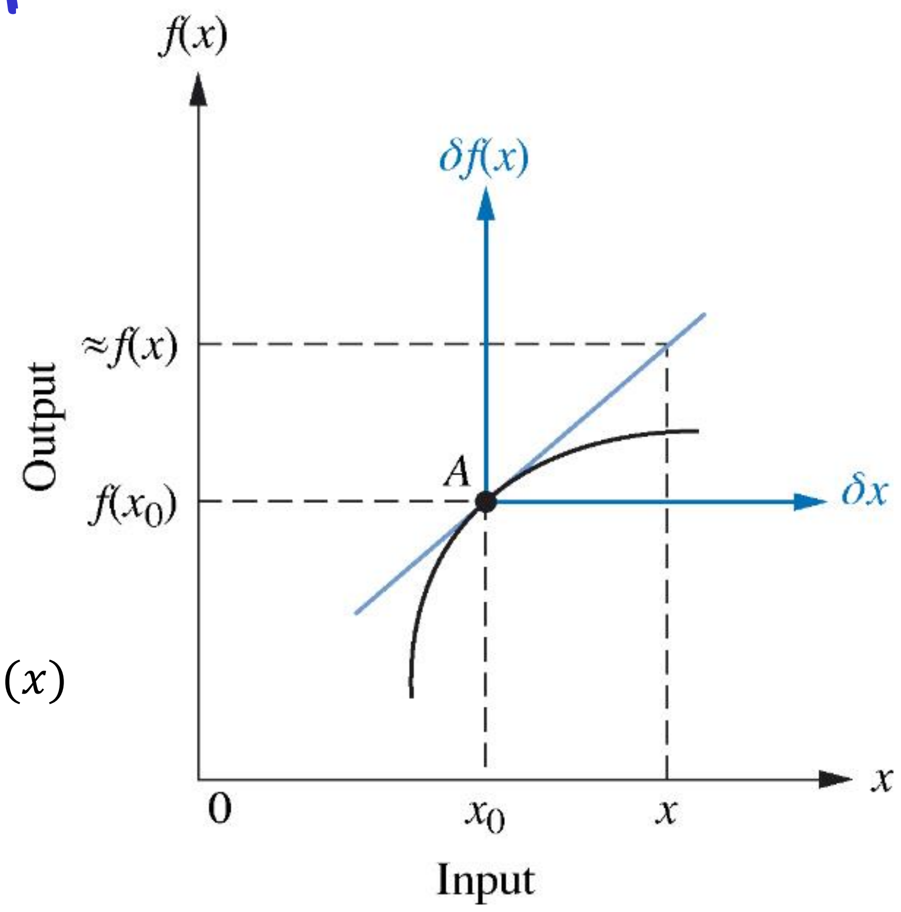
small changes in the input δx → small changes in the output $\delta f(x)$

Thus, $f(x) - f(x_0) \approx m_A(x - x_0)$

→ $\delta f(x) \approx m_A \, \delta x$ — Derivative of $f(x)$ at $x = x_0$

→ $f(x) \approx f(x_0) + m_A(x - x_0) = f(x_0) + m_A \, \delta x$

→ $$f(x) \approx f(x_0) + m_A \, \delta x$$



Linearization about a point A

Linearizing a Function

Problem: Linearize $f(x) = 5\,cos(x)$ about $x = {}^{\pi}/_{2}$.

Solution:

We first find that the derivative of $f(x)$ at $x = {}^{\pi}/_{2}$

$$\left.\frac{df}{dx}\right|_{x=\pi/2} = -5\sin x\big|_{x=\pi/2} = -5$$

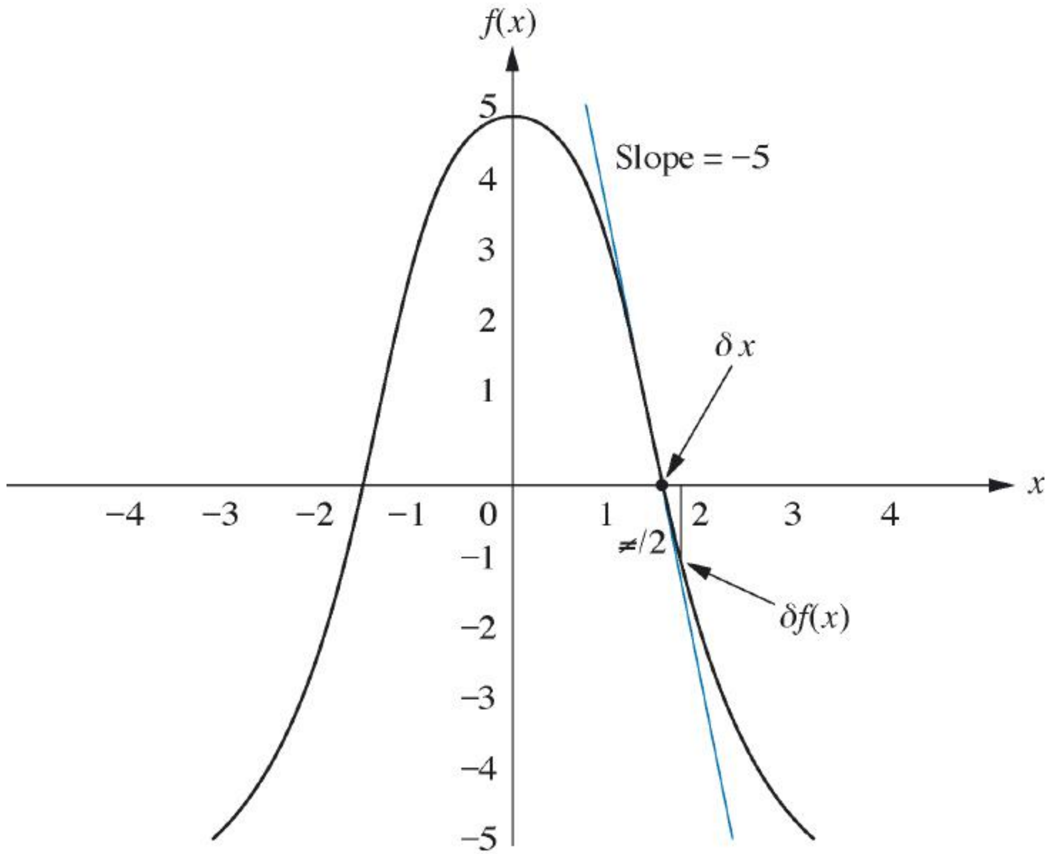
Slope at $x = \frac{\pi}{2}$

Also

$$f(x_0) = f(\pi/2) = 5\cos(\pi/2) = 0$$

the system can be represented as

$f(x) = -5\,\delta x$ for small excursions of x about ${}^{\pi}/_{2}$



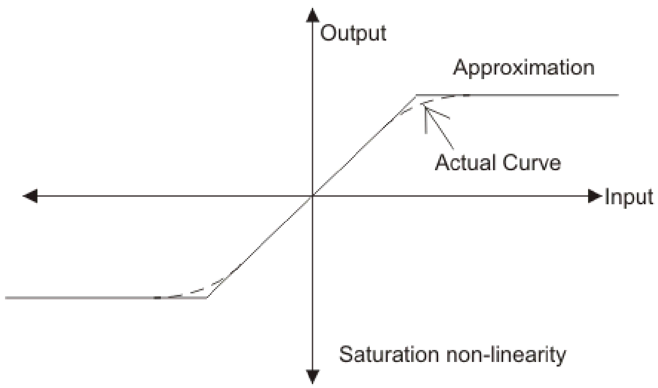
Modeling in The Time Domain

State-space Method

Two approaches are available for the analysis and design of feedback control systems.

1. *Frequency domain approach* (classical approach):

based on converting a system's differential equation to a transfer function.



- *Advantage*: rapidly providing stability and transient response information. Thus we can immediately see the effects of varying system parameters.

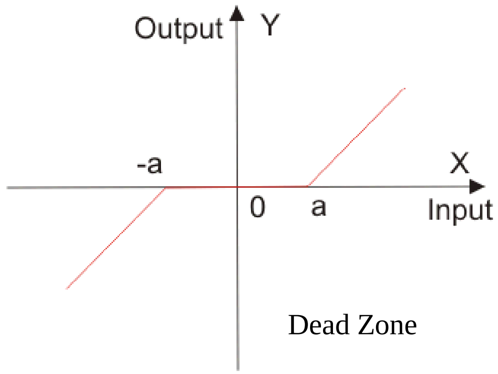
- *Disadvantage*: limited application. It can be applied only to linear, time-invarian systems or systems that can be approximated as such.

2. *State-space approach* (time domain / modern approach):

a sudden, forceful backward movement

Can be used:

a) To represent non-linear systems that have backlash, saturation, dead zone.



b) It can handle systems with nonzero initial conditions.

c) Multiple-inputs, multiple-outputs systems can easily be represented.

d) Many commercial software packages are available.

Many calculation is needed before actual realization.

RL Network: State-Space Representation

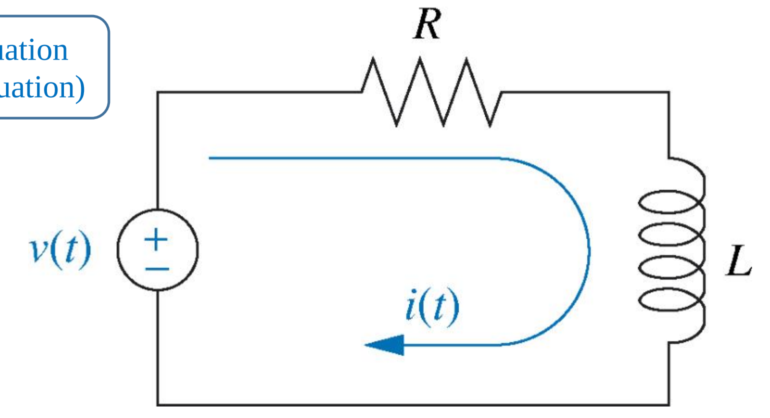
The *state-space* approach for representing physical systems (state equations and the output equations are a viable (feasible) representation of the system.).

1. Select a *state variable* (possible system variable) : say $i(t)$.

2. Write differential equation (in terms of the state variable $i(t)$).

$$L\frac{di}{dt}+Ri=v(t)$$

loop equation (State Equation)



RL network

3. Take Laplace transform:

solve for $I(s)$: $L[sI(s)-i(0)]+RI(s)=V(s)$ ➡ $I(s)=\dfrac{V(s)}{(Ls+R)}+\dfrac{L\,i(0)}{Ls+R}$

We can determine the state variable

If $v(t)=u(t), then\ V(s)={}^{1}/_{s}.$ ➡ $I(s)=\dfrac{1}{s(Ls+R)}+\dfrac{L\,i(0)}{Ls+R}$

➡ $I(s)=\dfrac{A}{s}+\dfrac{B}{Ls+R}+\dfrac{L\,i(0)}{Ls+R}$ ➡ $I(s)=\dfrac{1}{R}\left(\dfrac{1}{s}-\dfrac{1}{s+\frac{R}{L}}\right)+\dfrac{i(0)}{s+\frac{R}{L}}$

Inverse Laplace transform: $i(t)=\dfrac{1}{R}\left(1-e^{-(R/L)t}\right)+i(0)e^{-(R/L)t}$

4. Output equations:

Algebraically combine the state variables with the system's input and find all of the other system variables for $t\geq t_0$.

$$v_R(t)=Ri(t)$$
$$v_L(t)=v(t)-Ri(t)$$
$$\frac{di}{dt}=\frac{1}{L}[v(t)-Ri(t)]$$

If we know initial condition of i, $i(0)$, and input voltage, $v(t)$, then we can find the value of any network variables at time $t\geq t_0$.

Self Study

Do the State-space representation of RLC network.

The General State-Space Representation

Some Terminology

- *Linear combination:* (of n variables x_i) $S = K_n x_n + K_{n-1} x_{n-1} + \ldots + K_1 x_1$

  none of the variables can be written as a linear combination of the others.

- *Linear independence:* S is zero if every K is zero and no x is zero: variables x are linearly independent.
- *System variable:* Any variable that responds to an input or initial conditions in a system.
- *State variables:* The smallest set of linearly independent system variables that completely determines (knowing the value at t_0) the value of system variables for $t \geq t_0$
- *State vector:* A vector whose elements are state variables.
- *State space:* The n-dimensional space whose axes are the state variables.
- *State equations:* A set of n simultaneous, first-order differential equations with n variables (state variables).
- *Output equations:* The equation that expresses the output variables of a system as linear combinations of the state variables and the inputs.

State-space Representation

- A system is represented in *state-space* by the following equations:

$$\begin{cases} \dot{x} = A\,x + B\,u & \leftarrow \textit{State equation} \\ y = C\,x + D\,u & \leftarrow \textit{Output equation} \end{cases}$$

x: *state vectot*
$\dot{x}$: *derivative of the state vector w.r.t. time*
y: *Output vector*
u: *input or controlvector*
A: *system matrix*
B: *input matrix*
C: *output matrix*
D: *feedforward matrix*

- This representation of a system provides *complete knowledge* of all variables of the system at any $t \geq t_0$

- The choice of state variables:
  - is not unique.
  - *minimum number* (equals the order of the differential equation).
  - are *linearly independent.*

Problem:

Given the following system:

Set the system on the following state-space form:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - x_2 + 5u$$

$$y = x_2$$

Solution:

State-space model:

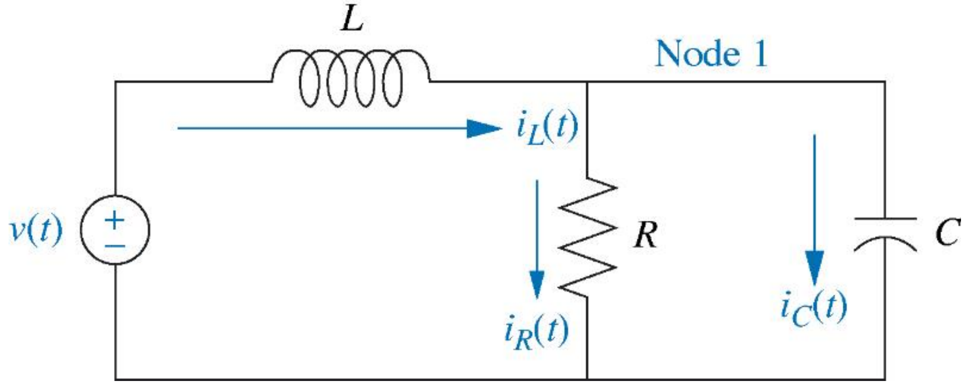
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 5 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{[0]}_{D} u$$

Example-1: State-space Representation

Problem:

Find a state-state representation of the following electrical network if the *output is i_R* the current through the resistor, (*$v(t)$ is the input*).



Solution:

The following steps will yield a viable representation of the network in state space.

Step 1: Label all the branch currents in the network (These include i_L, i_R, and i_C).

Step 2: Select the state variables (*quantities that are differentiated v_C* and i_L, energy-storage elements, the inductor C and the capacitor L) and write derivative equations.

$$C\frac{dv_C}{dt} = i_C, \qquad L\frac{di_L}{dt} = v_L$$

We can have $\dot{v_c} = \dot{i_L}$.

Step 3: Express *non-state variables* (right-hand side: i_C and v_L) as a linear combinations of the state variables (differentiated variables: $v_C$and i_L) and the input, $v(t)$.

We have $v_R = v_C$.

$$i_C = -i_R + i_L = -\frac{1}{R}v_C + i_L$$

$$v_L = -v_C + v(t)$$

Apply Kirchhoff's voltage and current laws, to obtain i_C and v_L in terms of the state variables, v_C and i_L.

Example-1: State-space Representation-contd.

Step 4: Obtain state equations: (by substituting the values and rearranging)

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L, \qquad L\frac{di_L}{dt} = -v_C + v(t)$$

$$\Rightarrow \frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L$$

$$\frac{di_L}{dt} = -\frac{1}{L}v_C \qquad + \frac{1}{L}v(t)$$

Matrix Form

$$\begin{cases} \dfrac{dv_C}{dt} = -\dfrac{1}{RC}\cdot v_C + \dfrac{1}{C}\cdot i_L + 0\cdot v(t) \\ \dfrac{di_L}{dt} = -\dfrac{1}{L}\cdot v_C + 0\cdot i_L + \dfrac{1}{L}\cdot v(t) \end{cases}$$

State equation

Step 5: Find the output equation: $i_R = \frac{1}{R}v_C$

Matrix Form

$$i_R = \frac{1}{L}\cdot v_C + 0\cdot i_L$$

output equation

Final result: Convert into vector-matrix form

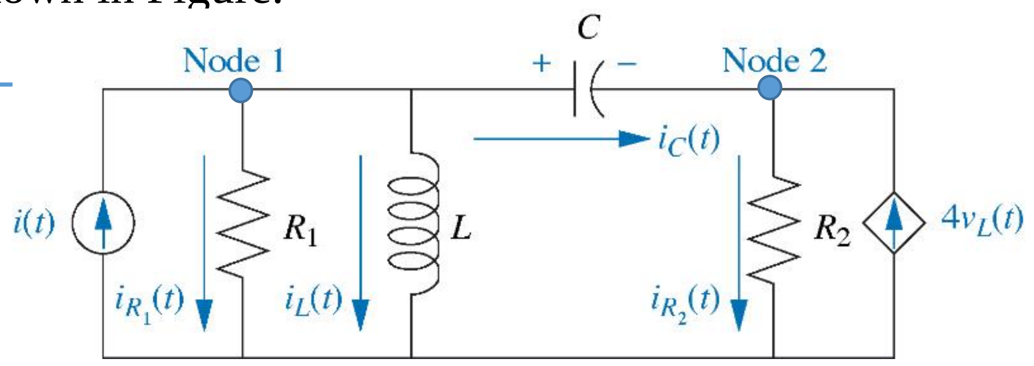
$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

Example-2: State-space Representation
(with a dependent source)

PROBLEM: Find the state and output equations for the electrical network shown in Figure.
If the output vector is $y = [v_{R_2} \quad i_{R_2}]^T$



Step 1: Label all the branch currents in the network.

Step 2: Select the *state variables* (energy-storage elements: L and C) and write derivative equations (voltage-current relationships).

$$C\frac{dv_C}{dt} = i_C, \qquad L\frac{di_L}{dt} = v_L \qquad x_1 = i_L; \quad x_2 = v_C;$$ the state variables (differentiated variables)

Step 3: State equations (we find v_L and i_C in terms of the state variables)

mesh LCR_2 → $v_L = v_C + v_{R2} = v_C + i_{R2}R_2$

Node 2 → At node 2, $i_{R2} = i_C + 4v_L$, so we get,

$$v_L = v_C + (i_C + 4v_L)R_2$$

$$\Rightarrow v_L = \frac{1}{1-4R_2}(v_C + i_C R_2)$$

$$(1 - 4R_2)v_L - R_2 i_C = v_C \qquad (1)$$

Node 1

$$i_C = i(t) - i_{R1} - i_L$$

$$= i(t) - \frac{v_{R1}}{R_1} - i_L$$

$(v_{R_1} = v_L)$

$$= i(t) - \frac{v_L}{R_1} - i_L$$

$$-\frac{1}{R_1}v_L - i_C = i_L - i(t) \qquad (2)$$

Example-2: State-space Representation

Solving (1) and (2) simultaneously for v_L and i_C yields

$$-\frac{1}{R_1}v_L - i_C = i_L - i(t) \Rightarrow i_C = -\frac{1}{R_1}v_L - i_L + i(t) \overset{(1)}{\Rightarrow} (1-4R_2)v_L - R_2(-\frac{1}{R_1}v_L - i_L + i(t)) = v_C$$

$$\Rightarrow \left(1 - 4R_2 + \frac{R_2}{R_1}\right)v_L + R_2 i_L - R_2 i(t) = v_C \Rightarrow v_L = \frac{1}{\Delta}[R_2 i_L - v_C - R_2 i(t)] \quad with\ \Delta = -\left(1 - 4R_2 + \frac{R_2}{R_1}\right)$$

and $$i_C = \frac{1}{\Delta}\left[(1-4R_2)i_L + \frac{1}{R_1}v_C - (1-4R_2)i(t)\right]$$

writing the result in vector-matrix form

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} R_2/(L\Delta) & -1/(L\Delta) \\ (1-4R_2)/(C\Delta) & 1/(R_1 C\Delta) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/(L\Delta) \\ -(1-4R_2)/(C\Delta) \end{bmatrix} i(t)$$

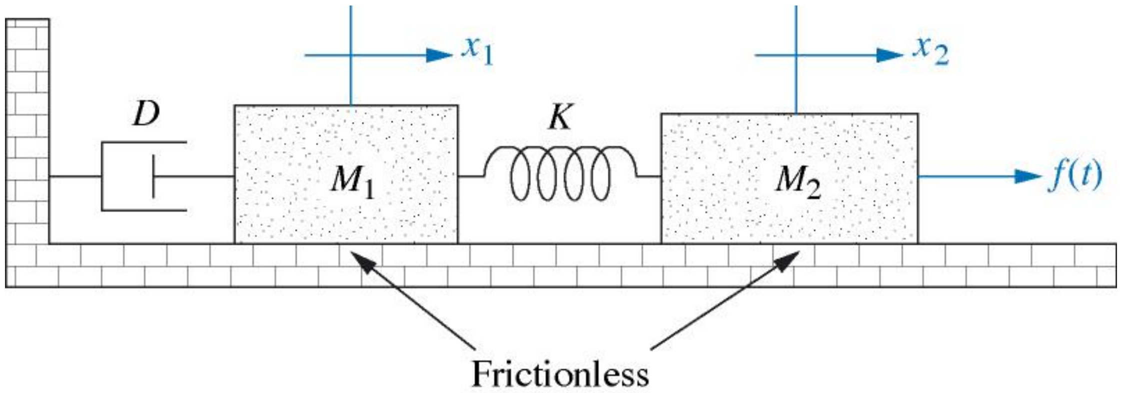
Step 4: Output equations $v_{R2} = -v_C + v_L$; $i_{R2} = i_C + 4v_L$;

vector-matrix form, the output equation is

$$\begin{bmatrix} v_{R_2} \\ i_{R_2} \end{bmatrix} = \begin{bmatrix} R_2/\Delta & -(1+1/\Delta) \\ 1/\Delta & (1-4R_1)/(\Delta R_1) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/\Delta \\ -1/\Delta \end{bmatrix} i(t)$$

Example-3: State-space Representation

(Translational Mechanical System)



For M1: $M_1 s^2 X_1(s) + DsX_1(s) + KX_1(s) - KX_2(s) = 0$

$$\Rightarrow M_1 \frac{d^2 x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

For M2:

$$-KX_1(s) + KX_2(s) + M_2 s^2 X_2(s) = F(s)$$

$$\Rightarrow -Kx_1 + Kx_2 + M_2 \frac{d^2 x_2}{dt^2} = f(t)$$

Let, $\dfrac{d^2 x_i}{dt^2} = \dfrac{dv_i}{dt}$

(acceleration = derivative of velocity)

Select x_1, x_2, v_1, v_2 as *state variables*.

$$\frac{dx_1}{dt} = v_1 \;\Rightarrow\; \frac{d^2 x_1}{dt^2} = \frac{dv_1}{dt} = \dot{v}_1$$

$$\frac{dx_2}{dt} = v_2 \;\Rightarrow\; \frac{d^2 x_2}{dt^2} = \frac{dv_2}{dt} = \dot{v}_2$$

State equations:

$$\frac{dx_1}{dt} = +v_1$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$

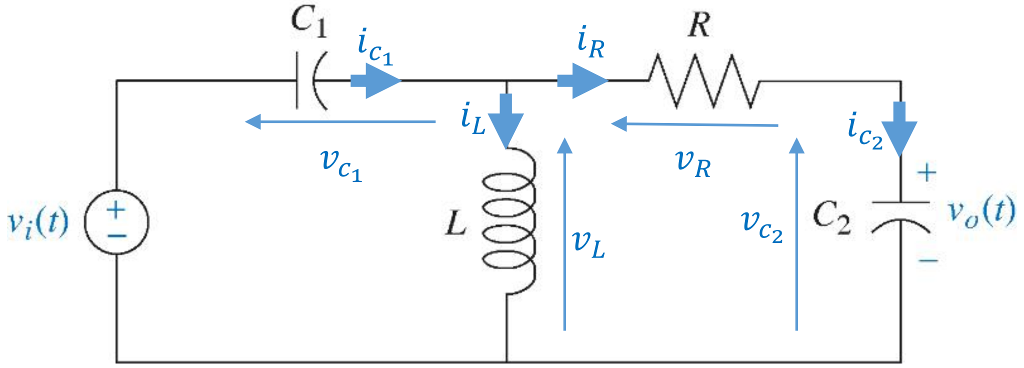
$$\frac{dx_2}{dt} = +v_2$$

$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t)$$

matrix form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & K/M_1 & -K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

Example-4: State-space Representation



Problem: Find the state-space representation of the electrical network shown in the figure. The output is $v_0(t)$.

Solution:

state variables: v_{c_1}, i_L, v_{c_2}

The derivative relations (one for each energy-storage element)

$$C_1\frac{dv_{C1}}{dt}=i_{C1}$$
$$L\frac{di_L}{dt}=v_L$$
$$C_2\frac{dv_{C2}}{dt}=i_{C2}$$

Using Kirchhoff's current and voltage laws:

$$i_{c_1} = i_L + i_R = i_L + \frac{v_R}{R} = i_L + \frac{1}{R}\left(v_L - v_{c_2}\right) \quad \textit{Mesh 2}$$
$$v_L = v_i - v_{c_1} \quad \textit{Mesh 1}$$
$$i_{c_2} = i_R = \frac{1}{R}\left(v_L - v_{c_2}\right) = \frac{1}{R}\left(v_i - v_{c_1} - v_{c_2}\right) \quad \textit{Mesh 2, Mesh 1}$$

$$i_{c_1} = -\frac{1}{R}v_{c_1} + i_L - \frac{1}{R}v_{c_2} + \frac{1}{R}v_i$$
$$v_L = -\,v_{c_1} + 0\,i_L + 0\,v_{c_2} + v_i$$
$$i_{c_2} = -\frac{1}{R}v_{c_1} + 0\,i_L - \frac{1}{R}v_{c_2} + \frac{1}{R}v_i$$

Matrix form

State vector

$$x = \begin{bmatrix} v_{c_1} \\ i_L \\ v_{c_2} \end{bmatrix}$$

state-space representation

$$\dot{x} = \begin{bmatrix} -{}^{1}/_{RC_1} & {}^{1}/_{C_1} & -{}^{1}/_{RC_1} \\ -{}^{1}/_{L} & 0 & 0 \\ -{}^{1}/_{RC_2} & 0 & -{}^{1}/_{RC_2} \end{bmatrix} x + \begin{bmatrix} {}^{1}/_{RC_1} \\ {}^{1}/_{L} \\ {}^{1}/_{RC_2} \end{bmatrix} v_i(t)$$

Converting a Transfer Function to State Space

Phase variables: A set of state variables where each state variable is defined to be the derivative of the previous state variable.

Consider a differential equation, $$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

Choose the output, y(t), and its derivatives as the state variables, x_i.

$$x_1 = y$$
$$x_2 = \frac{dy}{dt}$$
$$x_3 = \frac{d^2 y}{dt^2}$$
$$\cdot$$
$$x_n = \frac{d^{n-1} y}{dt^{n-1}}$$

Differentiating

$$\dot{x}_1 = \frac{dy}{dt}$$
$$\dot{x}_2 = \frac{d^2 y}{dt^2}$$
$$\dot{x}_3 = \frac{d^3 y}{dt^3}$$
$$\cdot$$
$$\dot{x}_n = \frac{d^n y}{dt^n}$$

Substituting

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\cdot$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_0 x_1 - a_1 x_2 \ldots - a_{n-1} x_n + b_0 u$$

What is output equation?

Converting a Transfer Function to State Space

matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

output

$$y = [1 \quad 0 \quad 0 \quad \dots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

PROBLEM: Find the state-space representation in phase-variable form for the transfer function

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

Step 1 Find the associated differential equation

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

inverse Laplace transform,

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Step 2 Select the state variables.

$$x_1 = c$$
$$x_2 = \dot{c}$$
$$x_3 = \ddot{c}$$

the state equations.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$
$$y = c = x_1$$

matrix form,

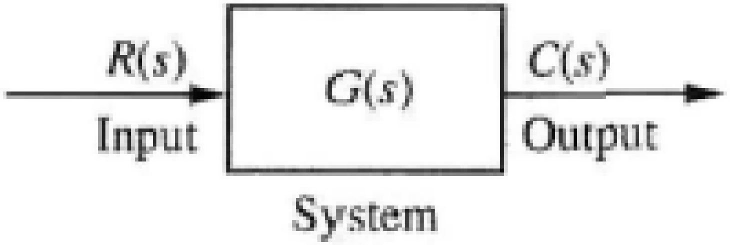
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

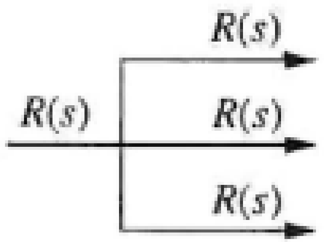
Block Diagram Reduction

- More complicated systems are represented by the interconnection of many subsystems.
- In order to calculate the transfer function, we want to represent multiple subsystems as a single block.
- A subsystem is represented as a block with an input, an output, and a transfer function.

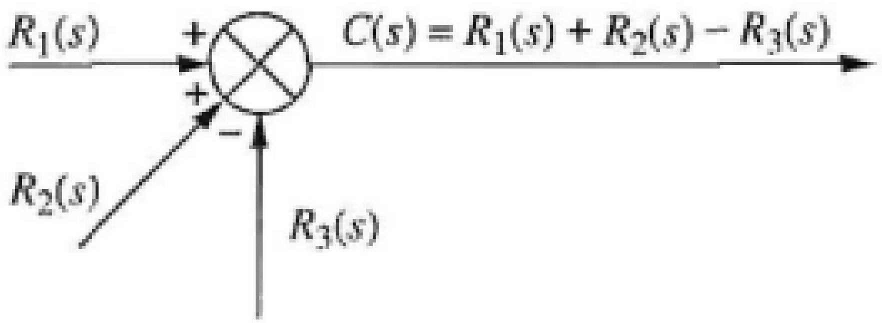
Signals



block diagram



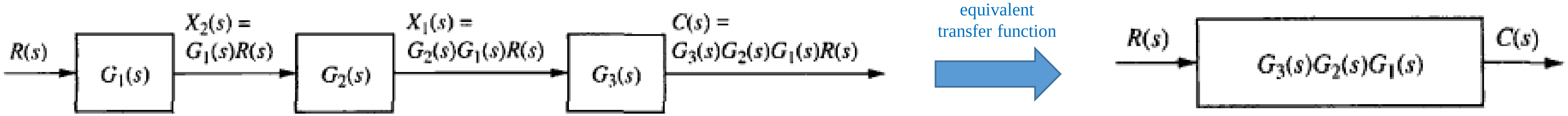
Pickoff point



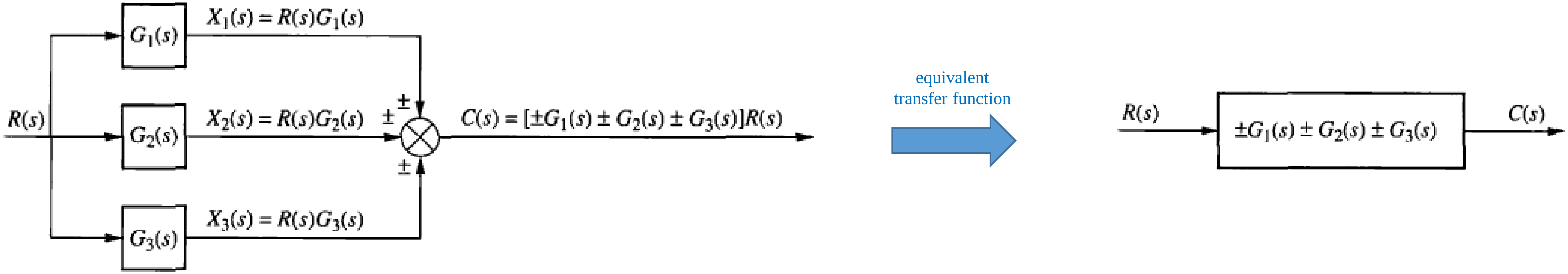
Summing junction

Reduction of Multiple Subsystems$_2$

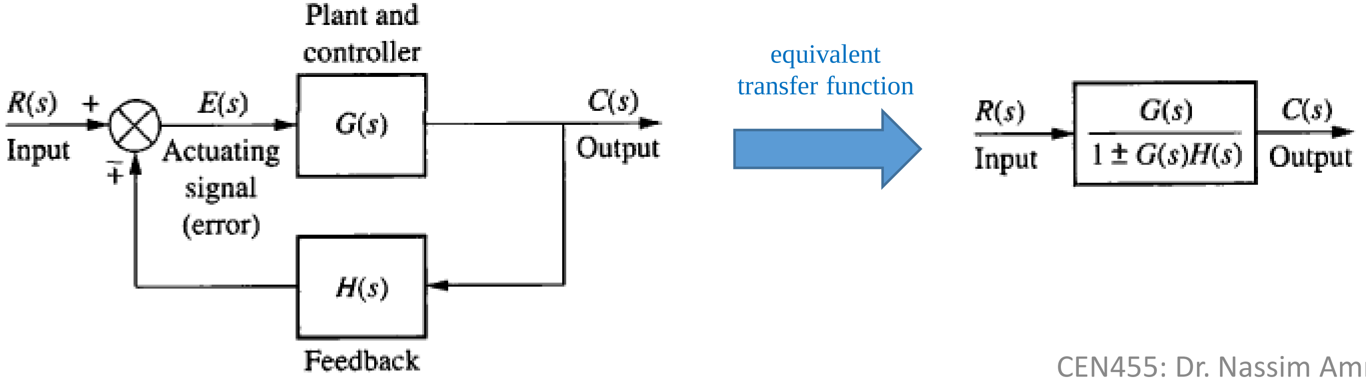
Cascade Form



Parallel Form



Feedback Form



$$C(s) = G(s)\,E(s)$$

But since $E(s) = R(s) \mp C(s)\,H(s)$

$$C(s) = G(s)\,[R(s) \mp C(s)\,H(s)\,]$$

$$C(s) = G(s)R(s) \mp G(s)C(s)\,H(s)$$

$$[1 \pm G(s)\,H(s)]C(s) = G(s)R(s)$$

equivalent transfer function

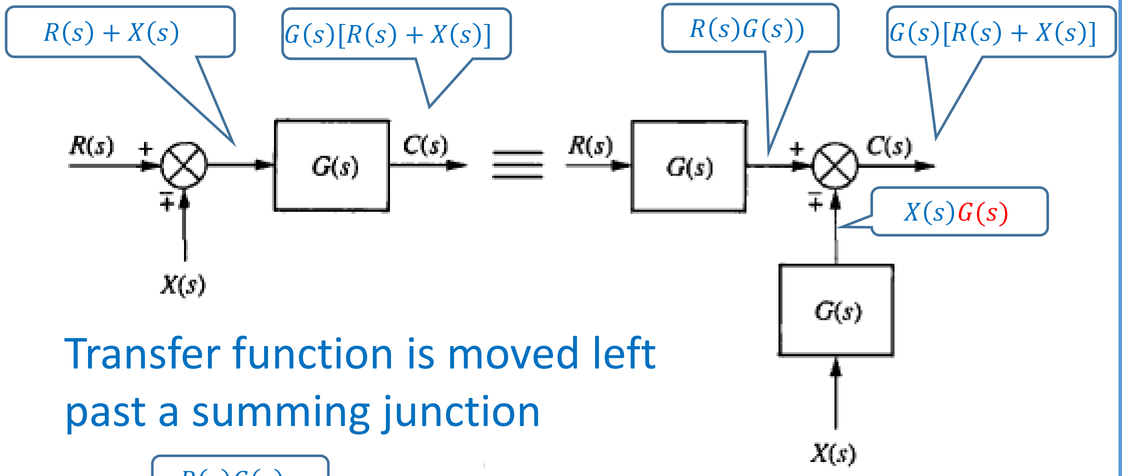
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)\,H(s)}$$

Reduction of Multiple Subsystems$_3$

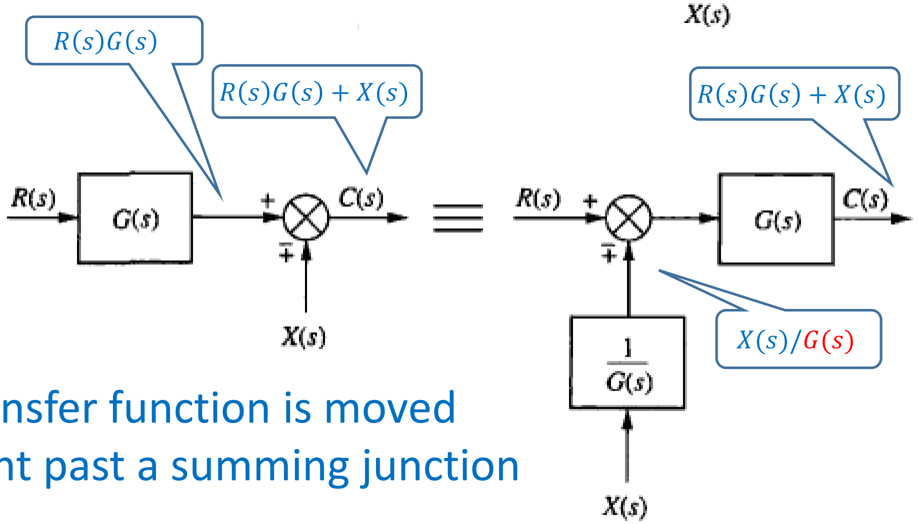
Moving Blocks to Create Familiar Forms

- Familiar forms (cascade, parallel, and feedback) are not always apparent in a block diagram

Block diagram algebra for summing junction

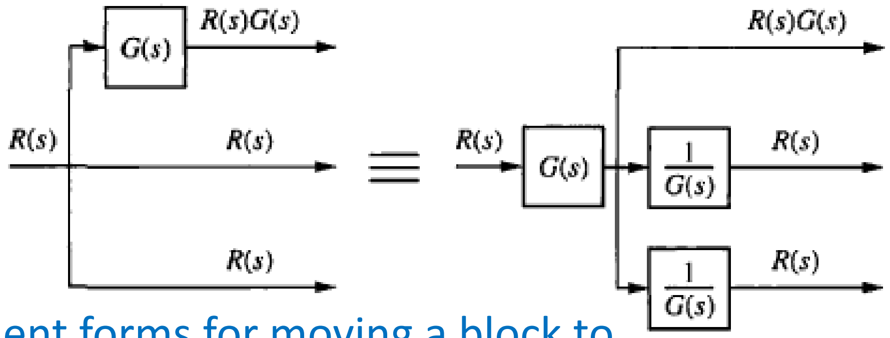


Transfer function is moved left past a summing junction

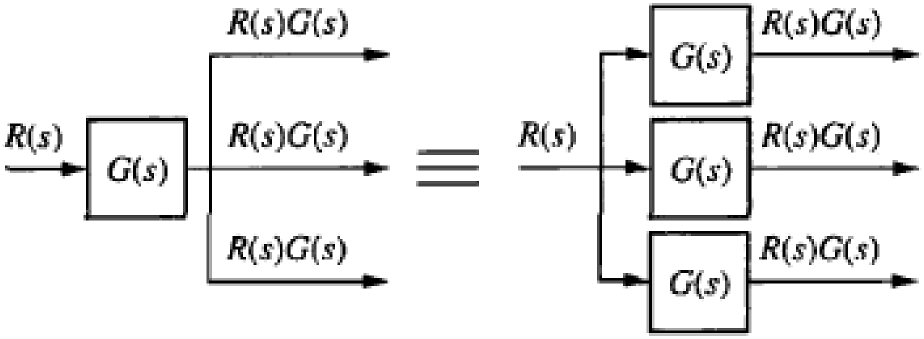


Transfer function is moved right past a summing junction

Block diagram algebra for pickoff point



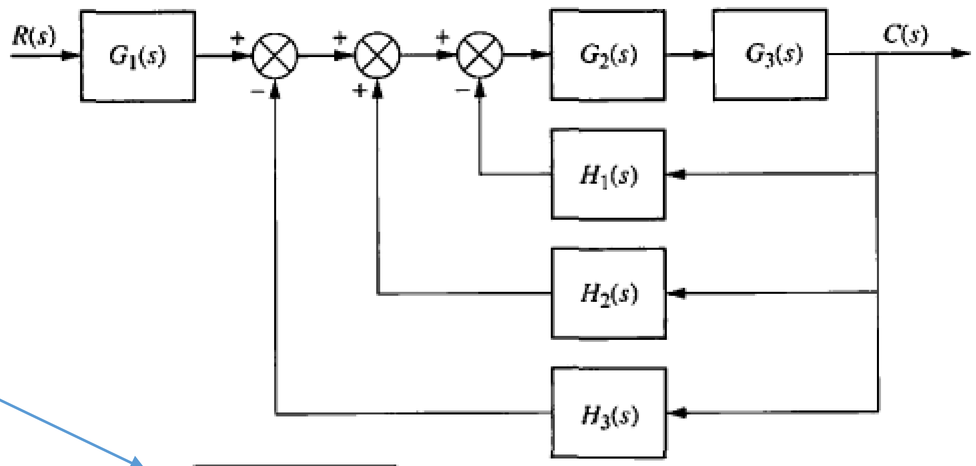
Equivalent forms for moving a block to the left past a pickoff point.



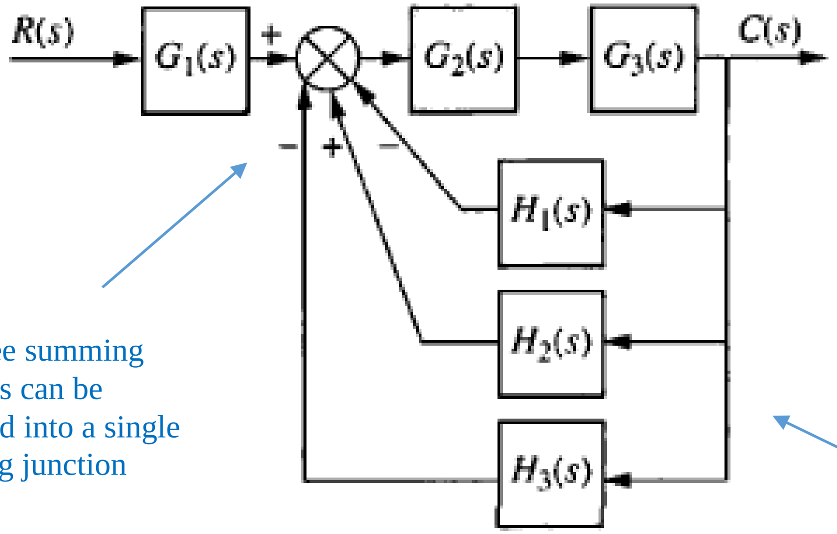
Equivalent forms for moving a block to the right past a pickoff point.

Example

Problem: Reduce the system shown in Figure to a single transfer function.

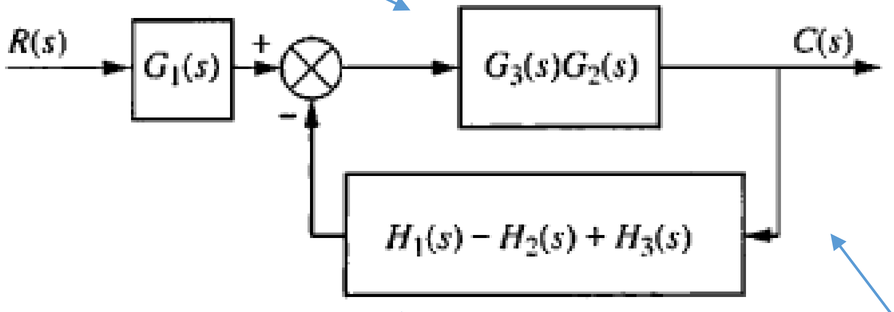


Solution:



The three summing junctions can be collapsed into a single summing junction

$G_2(s)$ *and* $G_3(s)$ are connected in cascade.



the three feedback functions, $H_1(s), H_2(s),$ *and* $H_3(s)$ are connected in parallel.

the feedback system is reduced and multiplied by $G_1(s)$