King Saud University
Department of Mathematics
M-203
(Differential and Integral Calculus)
Second Mid-Term Examination
(II-Semester 1431/1432)

Max. Marks: 20 Time: 90 Minutes

Marking Scheme: Q.1(4), Q.2:(3), Q.3:(3), Q.4:(3), Q.5:(3), Q.6:(4)

Q. No: 1 Reverse the order of integration, and evaluate the resulting integral

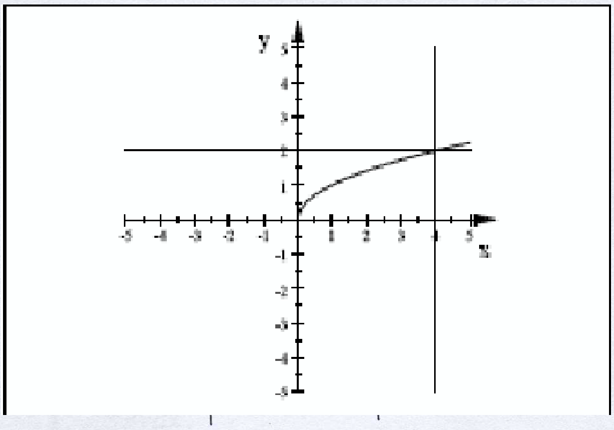
$$\int_0^4\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dydx.$$

Solution: Here $0 \le x \le 4$ and $\sqrt{x} \le y \le 2.$.

$\Rightarrow 0 \le y \le 2,\quad 0 \le x \le y^2.$

$$\Rightarrow \int_0^4\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dydx = \int_0^2\int_0^{y^2} \frac{1}{y^3+1}\,dxdy = \int_0^2 \frac{1}{y^3+1}\left[x\right]_0^{y^2} dy$$

$$= \int_0^2 \frac{y^2}{y^3+1}\,dy = \frac{1}{3}\left[\ln(y^3+1)\right]_0^2 = \frac{1}{3}\left[\ln(9)-\ln(1)\right] = \frac{1}{3}\ln(9).$$



Q. No: 2 Use **polar coordinates** to evaluate the integral

$\iint_R \left(x^2-y^2\right)dA$, where R is the region

bounded by the semi-circle $y=\sqrt{1-x^2}$ and the $x-axis$.

Solution: $\iint_R \left(x^2-y^2\right)dA = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2-y^2)dxdy = \int_0^{\pi}\int_0^1 (r^2\cos^2\theta - r^2\sin^2\theta)rdrd\theta$

$$= \int_0^{\pi} \int_0^1 (\cos^2\theta - \sin^2\theta) r^3 dr d\theta = \int_0^{\pi} \cos(2\theta)\left[\frac{r^4}{4}\right]_0^1 d\theta = \frac{1}{4}\int_0^{\pi} \cos(2\theta) d\theta$$

$$= \frac{1}{4}\left[\frac{\sin(2\theta)}{2}\right]_0^{\pi} = \frac{1}{8}[\sin(2\pi) - \sin(0)] = \frac{1}{8}[0] = 0$$

Q. No: 3 Find the **surface area** of the surface S if S is the portion of the graph of $z = 2 + xy$ that lies inside the **cylinder** $x^2 + y^2 = 1$.

Solution: **Surface area=**

$$\iint_{R_{xy}} \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint_{R_{xy}} \sqrt{1 + y^2 + x^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\, r dr d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left[\frac{(1+r^2)^{3/2}}{3/2}\right]_0^1 d\theta = \frac{1}{3}\left[(2)^{3/2} - 1\right]2\pi .$$

Q. No: 4 Set up integrals that can be used to find the centroid of the solid Q, where Q is **bounded** by the **co-ordinate planes,** and graphs of the equations

$z = 9 - x^2$ and $2x + y = 6$.

Solution: Region is $0 \le x \le 3, 0 \le y \le 6 - 2x, 0 \le z \le 9 - x^2$.

$$m = \int_0^3 \int_0^{6-2x} \int_0^{9-x^2} dz dy dx ,$$

$$M_{xy} = \int_0^3 \int_0^{6-2x} \int_0^{9-x^2} z\, dz dy dx , \quad M_{yz} = \int_0^3 \int_0^{6-2x} \int_0^{9-x^2} x\, dz dy dx ,$$

$$M_{xz} = \int_0^3 \int_0^{6-2x} \int_0^{9-x^2} y\, dz dy dx .$$

$\bar{x} = \frac{M_{yz}}{m}$ $\quad \bar{y} = \frac{M_{xz}}{m}$ $\quad \bar{z} = \frac{M_{xy}}{m}$

Q. No: 5 **Find the mass of the solid bounded by** $z = x^2 + y^2 - 4$ **and** $z = 0$ **having density** $\delta(x, y, z) = 1 + x^2 + y^2$.

Solution: Region is $x^2 + y^2 - 4 \le z \le 0, -\sqrt{4 - x^2} \le y \le +\sqrt{4 - x^2}, -2 \le x \le 2.$

In Cylindrical system region is $r^2 - 4 \le z \le 0, 0 \le r \le \sqrt{2}, 0 \le \theta \le 2\pi.$

Density $\delta(x,y,z)=1+x^2+y^2=1+r^2$.

$$Mass = \int_0^{2\pi}\int_0^{2}\int_{r^2-4}^{0}(1+r^2)rdzdrd\theta = \int_0^{2\pi}\int_0^{2}(r+r^3)[z]_{r^2-4}^{0}drd\theta$$

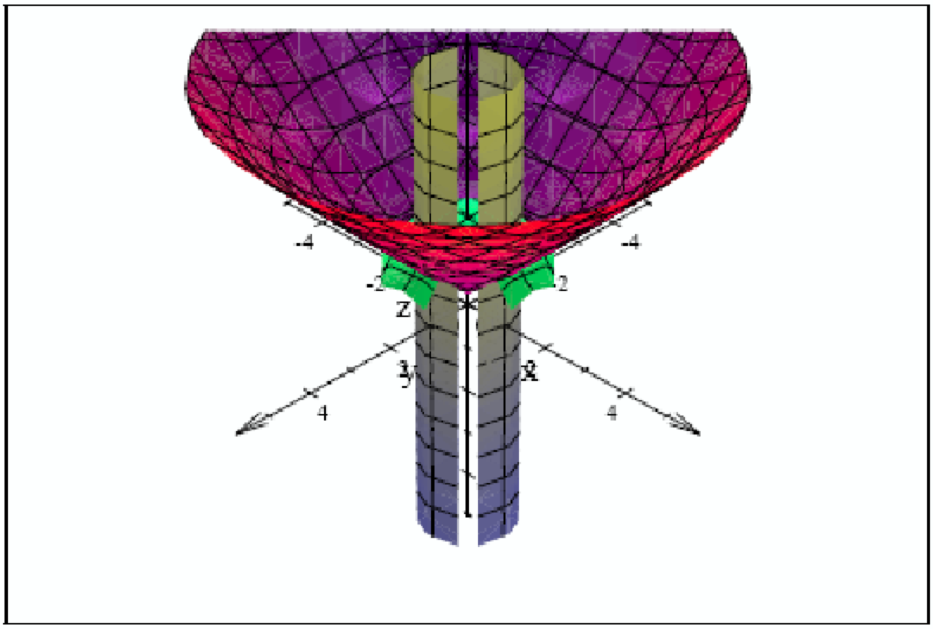
$$= \int_0^{2\pi}\int_0^{2}-(r+r^3)(r^2-4)drd\theta = \int_0^{2\pi}\int_0^{2}(+3r^3+4r-r^5)drd\theta$$

$$= \int_0^{2\pi}\left[+3\frac{r^4}{4}+4\frac{r^2}{2}-\frac{r^6}{6}\right]_0^2 d\theta = \int_0^{2\pi}\frac{28}{3}d\theta = \frac{56\pi}{3}.$$

Q. No: 6 Use spherical coordinates to evaluate the integral

$$I = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z\, dzdydx$$

Solution:



$$\begin{aligned} I &= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2}\rho^3\cos\varphi\sin\varphi d\rho d\varphi d\theta + \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\int_0^{\frac{1}{\sin\varphi}}\rho^3\cos\varphi\sin\varphi d\rho d\varphi d\theta \\ &= \pi + \frac{\pi}{2} \\ &= \frac{3\pi}{2}. \end{aligned}$$