Chapter 3: Random Time Series Model

*The random process $\{y_t\}$ is called **general liner process** if it's possible to express it in the form:*

$$y_t = \mu + \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \dots, \qquad t = 0, \pm1, \pm2, \dots$$

where, $\varepsilon_t \sim iid\ WN(0, \sigma_\varepsilon^2)$, μ is a constant and $\{\psi_t\}$ is a sequence of fixed values.

the process $\{y_t\}$ is stationary if one of the following conditions is satisfied:

1. *ψ_i are finite.*
2. *$\sum_{i=0}^{\infty} \psi_i^2 < \infty$.*

Question 1:

For the model $y_t = \varepsilon_t + 0.5y_{t-1}$ find π-weights and ψ-weights.

The π-weights formula $y_t = \varepsilon_t + \pi_1 y_{t-1} + \pi_2 y_{t-2} + \pi_3 y_{t-3} + \dots$

$$\pi_1 = 0.5 \qquad \pi_2 = 0 \qquad \pi_3 = 0$$

The ψ-weights formula $y_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \dots$

And for ψ weight, we find:

$$y_t = \varepsilon_t + 0.5y_{t-1}$$
$$y_{t-1} = \varepsilon_{t-1} + 0.5y_{t-2}$$
$$y_{t-2} = \varepsilon_{t-2} + 0.5y_{t-3}$$
$$y_{t-3} = \varepsilon_{t-3} + 0.5y_{t-4}$$

$$y_t = \varepsilon_t + 0.5[y_{t-1}]$$
$$y_t = \varepsilon_t + 0.5[\varepsilon_{t-1} + 0.5y_{t-2}]$$
$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2[y_{t-2}]$$
$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2[\varepsilon_{t-2} + 0.5y_{t-3}]$$
$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2\varepsilon_{t-2} + 0.5^3y_{t-3}$$
$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2\varepsilon_{t-2} + 0.5^3[y_{t-3}]$$
$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2\varepsilon_{t-2} + 0.5^3[\varepsilon_{t-3} + 0.5y_{t-4}]$$
$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2\varepsilon_{t-2} + 0.5^3\varepsilon_{t-3} + 0.5^4y_{t-4}$$

$$\psi_1 = 0.5 \qquad \psi_2 = 0.5^2 = 0.25 \qquad \psi_3 = 0.5^3 = 0.125$$

Question 2:

For the model $y_t = \varepsilon_t - 0.3\varepsilon_{t-1}$ *find the first three* π *weights and first three* ψ *weights.*

solution:

The ψ *weights formula* $y_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \ldots$

$$\psi_1 = -0.3 \qquad \psi_2 = 0 \qquad \psi_3 = 0$$

The π *weights formula* $y_t = \varepsilon_t + \pi_1 y_{t-1} + \pi_2 y_{t-2} + \pi_3 y_{t-3} + \ldots$

And for π *weight, we find:*

$$y_t = \varepsilon_t - 0.3\varepsilon_{t-1} \Rightarrow \quad \varepsilon_t = y_t + 0.3\varepsilon_{t-1}$$

$$\varepsilon_{t-1} = y_{t-1} + 0.3\varepsilon_{t-2}$$

$$\varepsilon_{t-2} = y_{t-2} + 0.3\varepsilon_{t-3}$$

$$\varepsilon_{t-3} = y_{t-3} + 0.3\varepsilon_{t-4}$$

$$\varepsilon_t = y_t + 0.3[\varepsilon_{t-1}]$$

$$\varepsilon_t = y_t + 0.3[y_{t-1} + 0.3\varepsilon_{t-2}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2\varepsilon_{t-2}$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2[\varepsilon_{t-2}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2[y_{t-2} + 0.3\varepsilon_{t-3}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2 y_{t-2} + 0.3^3\varepsilon_{t-3}$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2 y_{t-2} + 0.3^3[\varepsilon_{t-3}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2 y_{t-2} + 0.3^3[y_{t-3} + 0.3\varepsilon_{t-4}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2 y_{t-2} + 0.3^3 y_{t-3} + 0.3^4\varepsilon_{t-4}$$

$$y_t = \varepsilon_t - 0.3y_{t-1} - 0.3^2 y_{t-2} - 0.3^3 y_{t-3} - 0.3^4\varepsilon_{t-4}$$

$$\pi_1 = -0.3 \qquad \pi_2 = -0.3^2 \qquad \pi_3 = -0.3^3$$

- ***Finding π-weights and ψ-weights for mixed:***

The ψ- weights formula

$$\boxed{y_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \ \ldots, \qquad t = 0, \pm 1, \pm 2, \ldots}$$

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots$$

The π- weights formula

$$\boxed{y_t = \varepsilon_t + \pi_1 y_{t-1} + \pi_2 y_{t-2} + \pi_3 y_{t-3} + \ \ldots, \qquad t = 0, \pm 1, \pm 2, \ldots}$$

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \cdots$$

To find ψ- weights $\boxed{\phi(B)\,\psi(B) = \theta(B)}$

To find π- weights $\boxed{\theta(B)\,\pi(B) = \phi(B)}$

Question 3:

For the model $y_t = 0.7y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$

find ψ-weights and π-weights.

solution:

to find the ψ-weight and π-weights:

$$y_t = 0.7y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$$

$$y_t - 0.7y_{t-1} = \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$$

$$y_t - 0.7y_{t-1} = \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$$

$$y_t - 0.7By_t = \varepsilon_t + 0.6\,B\varepsilon_t - 0.4\,B^2\varepsilon_t$$

$$(1 - 0.7B)y_t = (1 + 0.6\,B - 0.4\,B^2)\varepsilon_t$$

$$\phi(B) \quad y_t = \quad \theta(B) \quad \varepsilon_t$$

$$\textit{To find } \psi\textit{- weights } \boxed{\phi(B)\,\psi(B) = \theta(B)}$$

$$(1 - 0.7B)\,(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) = (1 + 0.6\,B - 0.4\,B^2)$$

B^0: $\boxed{\psi_0 = 1}$

$$(1 - 0.7B)\,(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) = (1 + 0.6\,B - 0.4\,B^2)$$

B^1: $\psi_1 - 0.7\psi_0 = 0.6 \implies \boxed{\psi_1 = 1.3}$

$$(1 - 0.7B)\,(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) = (1 + 0.6\,B - 0.4\,B^2)$$

B^2: $\psi_2 - 0.7\psi_1 = -0.4 \implies \boxed{\psi_2 = 0.3}$

$$(1 - 0.7B)\,(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) = (1 + 0.6\,B - 0.4\,B^2)$$

B^3: $\psi_3 - 0.7\psi_2 = 0 \implies \boxed{\psi_3 = 0.21}$

$$(1 - 0.7B)\,(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) = (1 + 0.6\,B - 0.4\,B^2)$$

B^4: $\psi_4 - 0.7\psi_3 = 0$

B^5: $\psi_5 - 0.7\psi_4 = 0$

$$\psi\textit{- weights} = \begin{cases} \psi_0 = 1 \\ \psi_1 = 1.3 \\ \psi_2 = 0.3 \\ \psi_j - 0.7\psi_{j-1} = 0 \quad ; \quad j \geq 3 \end{cases}$$

$$\textit{To find } \pi\text{- weights } \boxed{\theta(B)\,\pi(B) = \phi(B)}$$

$$(1 + 0.6\,B - 0.4\,B^2)\,(\pi_0 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \cdots) = (1 - 0.7B)$$

B^0: $\boxed{\pi_0 = 1}$

B^1: $\pi_1 + 0.6\,\pi_0 = -0.7 \implies \boxed{\pi_1 = -1.3}$

B^2: $\pi_2 + 0.6\,\pi_1 - 0.4\,\pi_0 = 0$

B^3: $\pi_3 + 0.6\,\pi_2 - 0.4\,\pi_1 = 0$

$$\pi\text{- weights} = \begin{cases} \pi_0 = 1 \\ \pi_1 = -1.3 \\ \pi_i + 0.6\,\pi_{i-1} - 0.4\,\pi_{i-2} = 0 \quad ; \quad i \geq 2 \end{cases}$$

Question 4:

For the model $y_t = 0.1y_{t-1} + \varepsilon_t + 0.2\varepsilon_{t-1}$

find ψ*-weights and* π*-weights.*

to find the ψ*-weight and* π*-weights:*

$$y_t = 0.1y_{t-1} + \varepsilon_t + 0.2\varepsilon_{t-1}$$

$$y_t - 0.1y_{t-1} = \varepsilon_t + 0.2\varepsilon_{t-1}$$

$$(1 - 0.1B)y_t = (1 + 0.2\,B)\varepsilon_t$$

$$\phi(B) \quad y_t = \quad \theta(B) \quad \varepsilon_t$$

- *To find* ψ*- weights* $\boxed{\phi(B)\,\psi(B) = \theta(B)}$

$$(1 - 0.1B)\,(\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots) = (1 + 0.2\,B)$$

B^0: $\boxed{\psi_0 = 1}$

B^1: $\psi_1 - 0.1\psi_0 = 0.2 \implies \boxed{\psi_1 = 0.3}$

B^2: $\psi_2 - 0.1\psi_1 = 0$

$$\psi\text{- weights} = \begin{cases} \psi_0 = 1 \\ \psi_1 = 0.3 \\ \psi_j - 0.1\psi_{j-1} = 0 \quad ; \quad j \geq 2 \end{cases}$$

- *To find* π*- weights* $\boxed{\theta(B)\,\pi(B) = \phi(B)}$

$$(1 + 0.2\,B)\,(\pi_0 + \pi_1 B + \pi_2 B^2 + \cdots) = (1 - 0.1B)$$

B^0: $\boxed{\pi_0 = 1}$

B^1: $\pi_1 + 0.2\,\pi_0 = -0.1 \implies \boxed{\pi_1 = -0.3}$

B^2: $\pi_2 + 0.2\,\pi_1 = 0$

$$\pi\text{- weights} = \begin{cases} \pi_0 = 1 \\ \pi_1 = -0.3 \\ \pi_i + 0.2\,\pi_{i-1} = 0 \quad ; \quad i \geq 1 \end{cases}$$

- *Auto-regressive processes:*

$$AR(p): \quad y_t = \varepsilon_t + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p}$$

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + +\phi_p \rho(k-p) \ ; \quad k = 1,2, \ldots$$

$$AR\ (1): y_t = \varepsilon_t + \phi_1 y_{t-1}$$

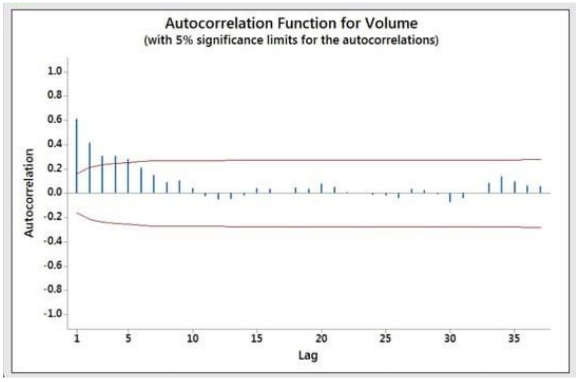
AR models are always invertible.

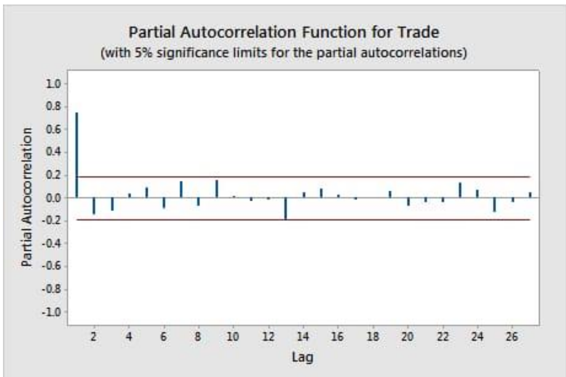
Stationary condition for AR (1):

$|\phi_1| < 1 \quad$ *or* $\quad |B| > 1$

Auto-correlation function (ACF) for AR (1):

$$\rho(k) = \phi^k \ ; \quad k = 1,2, \ldots$$





Question 5:

Assume the process $\{y_t\}$ *follow the AR (1) model with parameter* $\phi_1 = 0.4$, *and assuming that* $E(y_t) = 0$

1. *write down the mathematical form of this model.*

$$y_t = \varepsilon_t + 0.4y_{t-1},$$

where $\varepsilon_t \sim iid\ N(0, \sigma_\varepsilon^2)$ *is the white noise process*

2. *is* $\{y_t\}$ *stationary, why?*

the stationary condition for the AR (1) is:

$$|\phi_1| < 1 \Rightarrow |0.4| < 1$$

so, this process is stationary.

3. *Drive the autocorrelation function* ρ_k

$$y_t = \varepsilon_t + 0.4y_{t-1},$$

multiply both sides by y_{t-k} *and taking the expectation.*

$y_t y_{t-k} = \varepsilon_t y_{t-k} + 0.4y_{t-1}y_{t-k}$

$E(y_t y_{t-k}) = \underbrace{E(\varepsilon_t y_{t-k})}_{0} + 0.4E(y_{t-1}y_{t-k})$

$\gamma_k = 0.4\gamma_{k-1} \quad for\ \ k \geq 1$

dividing both sides by γ_0

$$\frac{\gamma_k}{\gamma_0} = 0.4\frac{\gamma_{k-1}}{\gamma_0}$$

$$\rho_k = 0.4\rho_{k-1} \quad for\ \ k \geq 1$$

$\rho_1 = 0.4\rho_0 = 0.4(1) = 0.4$

$\rho_2 = 0.4\rho_1 = 0.4(0.4) = 0.4^2$

$\rho_3 = 0.4\rho_2 = 0.4(0.4^2) = 0.4^3$

$\rho_k = 0.4\rho_{k-1} = 0.4(0.4^{k-1}) = 0.4^k$

$$AR\ (2):\ y_t = \varepsilon_t + \phi_1 y_{t-1} + \phi_2 y_{t-2}$$

AR models are always invertible.

Stationary condition for AR (2):

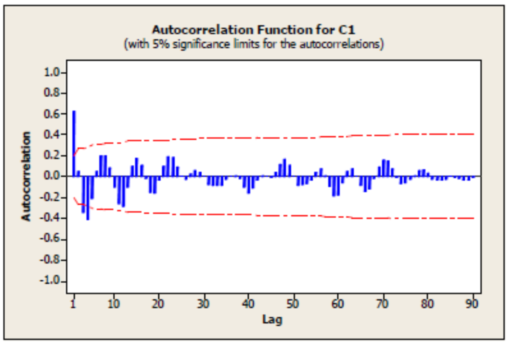
$|\phi_2| < 1$

$\phi_2 + \phi_1 < 1$

$\phi_2 - \phi_1 < 1$

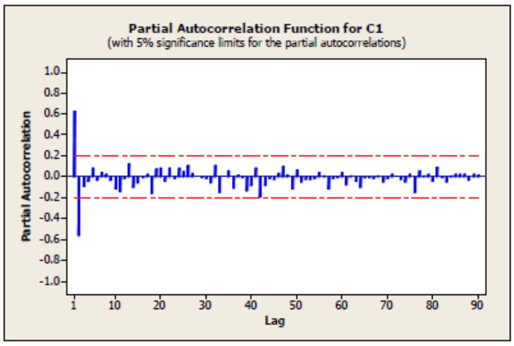
Auto-correlation function (ACF) for AR (2):

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2)\ ;\quad k = 1,2,\ldots$$



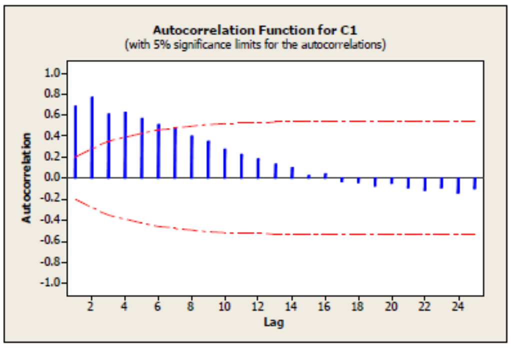
ACF for AR(2) model when

$\phi_1 = 1, \phi_2 = -0.5$



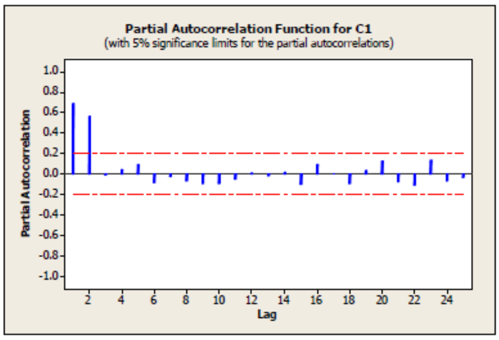
PACF for AR(2) model when

$\phi_1 = 1, \phi_2 = -0.5$



ACF for AR(2) model when

$\phi_1 = 0.4, \phi_2 = 0.5$



PACF for AR(2) model when

$\phi_1 = 0.4, \phi_2 = 0.5$

Question 6:

In the following question assume $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

For the following model, state its type, and check its stationarity and/or invertibility:

$$y_t = 0.7y_{t-1} + 0.5y_{t-2} + \varepsilon_t$$

model type: AR (2) or ARMA (2,0) or ARIMA (2,0,0)

stationarity: since $|\phi_2|$ *=|0.5| <1*

$\phi_2 - \phi_1$ *= 0.5-0.7=-0.2 <1*

$\phi_2 + \phi_1$ *= 0.5+0.7=1.2 >1*

then this model is not stationarity

invertibility: AR models are always invertible

- *Moving average:*

$$MA(q)\ \ y_t = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \cdots - \theta_q\varepsilon_{t-q}$$

$$\rho(k) = \begin{cases} \dfrac{-\theta_k+\theta_1\theta_{k+1}+\cdots+\theta_{q-k}\theta_q}{(1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2)} & ;\ \ k = 1,2,\ldots,q \\ 0 & ;\ \ k > q \end{cases}$$

$$MA\ (1):\ \ y_t = \varepsilon_t - \theta_1\varepsilon_{t-1}$$

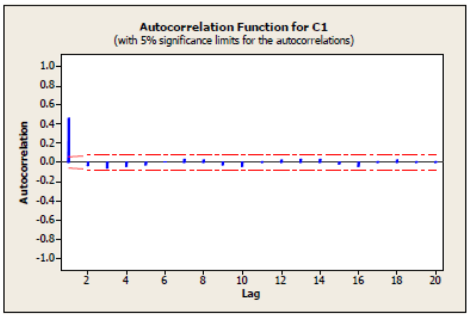
MA models are always stationary.

Invertibility conditions for the MA (1):

$|\theta_1| < 1$

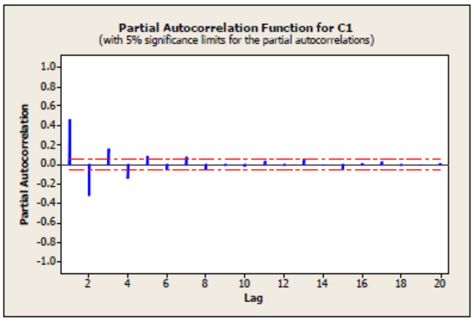
Auto-correlation function (ACF) for MA (1):

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\dfrac{\theta}{(1+\theta^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$



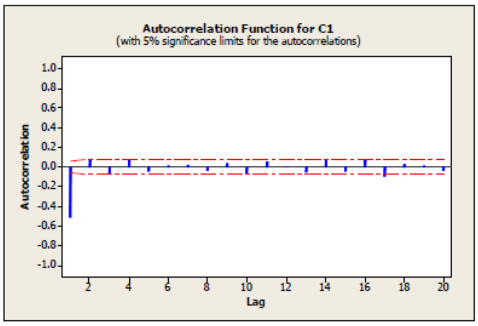
ACF for MA(1) model when

$\theta_1 = -0.7$



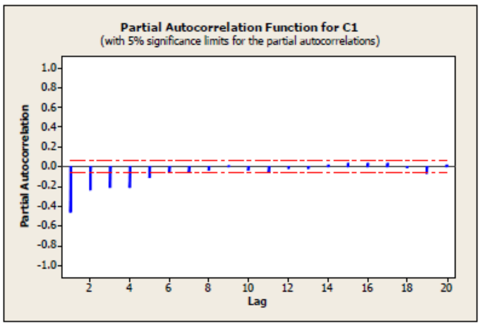
PACF for MA(1) model when

$\theta_1 = -0.7$



ACF for MA(1) model when

$\theta_1 = 0.7$



PACF for MA(1) model when

$\theta_1 = 0.7$

$$MA\ (2):\ y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

MA models are always stationary.

Invertibility conditions for the MA (2):

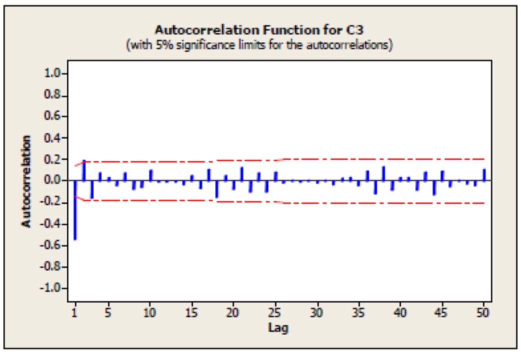
$|\theta_2| < 1$

$\theta_2 + \theta_1 < 1$

$\theta_2 - \theta_1 < 1$

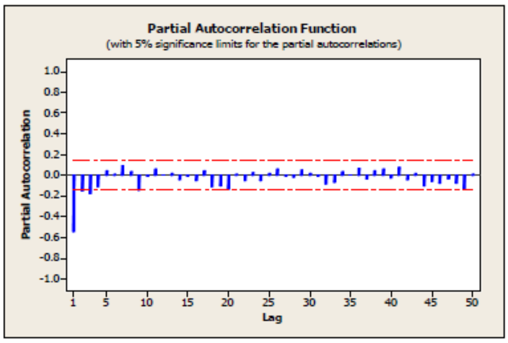
Auto-correlation function (ACF) for MA (2):

$$\rho(k) = \begin{cases} \frac{-\theta_1(1-\theta_2)}{(1+\theta_1^2+\theta_2^2)} & k = 1 \\ \frac{-\theta_2}{(1+\theta_1^2+\theta_2^2)} & k = 2 \\ 0 & k \geq 2 \end{cases}$$



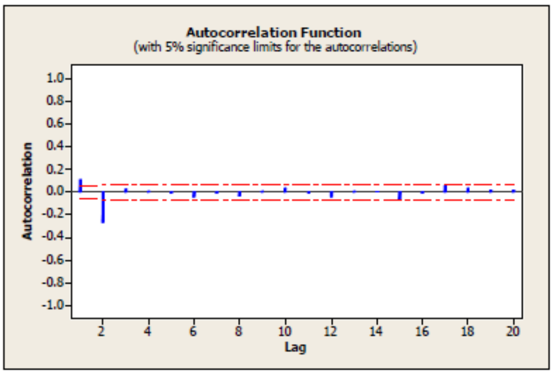
ACF for MA(2) model when

$\theta_1 = 0.7\,, \theta_2 = -0.1$



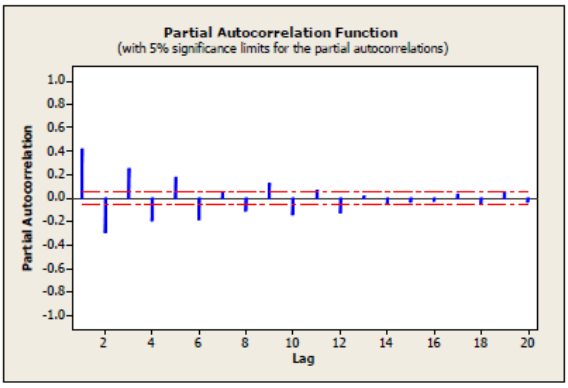
PACF for MA(2) model when

$\theta_1 = 0.7\,, \theta_2 = -0.1$



ACF for MA(2) model when

$\theta_1 = 1, \theta_2 = -0.7$



PACF for MA(2) model when

$\theta_1 = 1, \theta_2 = -0.7$

Question 7:

For MA (1) process with $\rho(1) = 0.4$,

Find the value of θ *that make this model is invertible.*

$$y_t = \varepsilon_t - \theta\ \varepsilon_{t-1}$$

$$Var(y_t) = Var(\varepsilon_t - \theta\ \varepsilon_{t-1}) = \sigma_\varepsilon^2 + \theta^2\ \sigma_\varepsilon^2 = (1+\theta^2\ )\sigma_\varepsilon^2$$

$$Cov(y_t, y_{t-k}) = E[(\varepsilon_t - \theta\ \varepsilon_{t-1})(\varepsilon_{t-k} - \theta\ \varepsilon_{t-k-1})]$$

$$for\ k = 1 \qquad E[(\varepsilon_t - \theta\ \varepsilon_{t-1})(\varepsilon_{t-1} - \theta\ \varepsilon_{t-2}\quad)] = -\theta\ \sigma_\varepsilon^2$$

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\frac{\theta}{(1+\theta^2\ )} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

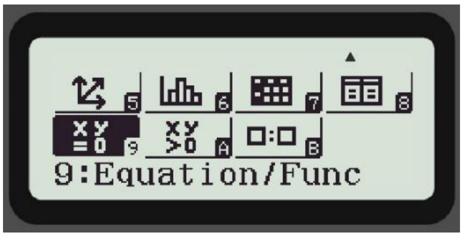
$$\rho(1) = \frac{\gamma_1}{\gamma_0} = \frac{-\theta\ \sigma_\varepsilon^2}{(1+\theta^2\ )\sigma_\varepsilon^2} = \frac{-\theta}{1+\theta^2} = 0.4$$

$$\frac{-\theta}{1+\theta^2} = 0.4 \Longrightarrow -\theta = 0.4 + 0.4\theta^2$$

$$\Longrightarrow 0.4\ \theta^2 + \theta + 0.4 = 0$$

or we can use the calculator

1



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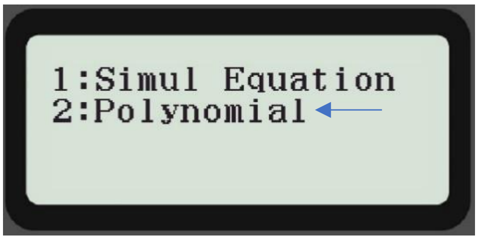

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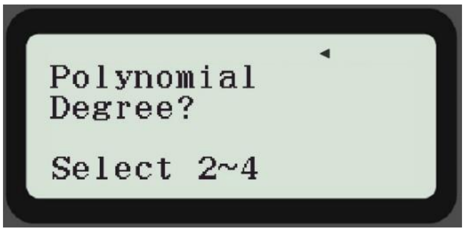

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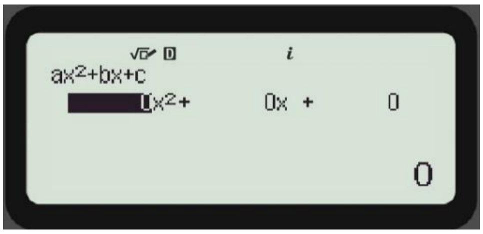

$$\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(0.4)(0.4)}}{2(0.4)} = (-0.5)\ or\ (-2)$$

MA (1) is invertibility $\Longrightarrow |\theta| < 1 \quad then,\ \theta = -0.5$

Question 8:

Assume the process $\{y_t\}$ *follow the MA (2) model with parameters*

$\theta_1 = 0.4 \; and \; \theta_2 = -0.8$

1. *write down the mathematical form of this model.*

$$y_t = \varepsilon_t - 0.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

where ε_t *is* $WN(0, \sigma_\varepsilon^2)$

2. *is* $\{y_t\}$ *invertible?*

the invertible condition for the MA (2) is:

- $|\theta_2| < 1 \quad \Rightarrow |-0.8| < 1$
- $\theta_2 + \theta_1 < 1 \Rightarrow -0.8 + 0.4 \Rightarrow -0.4 < 1$
- $\theta_2 - \theta_1 < 1 \Rightarrow -0.8 - 0.4 \Rightarrow -1.2 < 1$

the invertibility conditions are satisfied.

Question 9:

In the following question assume $\varepsilon_t \sim WN(0 \, , \sigma_\varepsilon^2)$

For the following model, ***state its type****, and check its* ***stationarity and/or invertibility****:*

$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} - 1.2\varepsilon_{t-2}$

model type: MA (2) or ARMA (0,2) or ARIMA (0,0,2)

Stationarity: Moving Average models are always stationary

Invertibility: since θ_2 *=1.2 >1, then this model is not invertible.*

Autoregressive-Moving average processes:

$$ARMA(p,q)\;\; y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}$$

for $ARMA(p,q)$:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}$$

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \cdots - \phi_p y_{t-p} = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}$$

$$y_t - \phi_1 B y_t - \phi_2 B^2 y_t - \cdots - \phi_p B^p y_t = \varepsilon_t - \theta_1 B \varepsilon_t - \theta_2 B^2 \varepsilon_t - \cdots - \theta_q B^q \varepsilon_t$$

$$\left(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p\right)\; y_t = \left(1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q\right)\; \varepsilon_t$$

$$\phi(B)\quad y_t = \theta(B)\quad \varepsilon_t$$

then, $ARIMA(p,d,q)$ *is:*

$$\phi(B)\; \nabla^d y_t = \theta(B)\; \varepsilon_t$$

$$\Rightarrow \phi(B)\quad z_t \quad = \theta(B)\; \varepsilon_t$$

$Model$	$ARMA(p,q)$	$ARIMA(p,d,q)$
$AR(p)$	$ARMA(p,0)$	$ARIMA(p,0,0)$
$AR(2)$	$ARMA(2,0)$	$ARIMA(2,0,0)$
$MA(q)$	$ARMA(0,q)$	$ARIMA(0,0,q)$
$MA(1)$	$ARMA(0,1)$	$ARIMA(0,0,1)$

$$ARMA(1,1)\ \ y_t = \phi_1 y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

Auto-correlation function (ACF) for $ARMA$ (1,1):

$$\rho_k = \begin{cases} \dfrac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{(1 + \theta_1^2 - 2\phi_1\theta_1)} & ;\ k = 1 \\ \phi_1 \rho_{k-1} & ;\ k = 2,3,\ldots \end{cases}$$

Question 10:

In the following question assume $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

For the following model, ***state its type****, and check its* ***stationarity and/or invertibility****:*

$$y_t = 1.5 y_{t-1} + \varepsilon_t + 0.25 \varepsilon_{t-1}$$

model type: ARMA (1,1) or ARIMA (1,0,1)

stationarity: since $|\phi_1| = |1.5| > 1$*, then this model is not stationarity*

invertibility: since $|\theta_1| = |-0.25| < 1$*, then this model is invertible.*

Question 11:

for the following model $y_t = 0.2\, y_{t-1} + \varepsilon_t - 1.1\, \varepsilon_{t-1} + 0.18\, \varepsilon_{t-2}$

$\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$*, state its type, and check its stationary and / or invertibility.*

$$y_t - 0.2\, y_{t-1} = \varepsilon_t - 1.1\, \varepsilon_{t-1} + 0.18\, \varepsilon_{t-2}$$

$$y_t - 0.2B\, y_t = \varepsilon_t - 1.1\, B\varepsilon_t + 0.18\, B^2 \varepsilon_t$$

$$(1 - 0.2B)y_t = (1 - 1.1\, B + 0.18\, B^2)\varepsilon_t$$

$$(1 - 0.2B)y_t = (1 - 0.2\, B)(1 - 0.9\, B)\varepsilon_t$$

$$y_t = (1 - 0.9\, B)\varepsilon_t$$

model type: MA (1) or ARMA (0,1) or ARIMA (0,0,1)

stationarity: MA (1) always stationarity

invertibility: since $|\theta_1| = |0.9| < 1$ *then this model is invertible.*

Question 12:

write the model $ARIMA(1,1,1)$ *in the final form in terms of*

$z_t = \nabla y_t$, y_t *and* ε_t

for $ARMA(1,1)$

$$y_t = \phi_1 y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - \phi_1 y_{t-1} = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - \phi_1 B y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$(1 - \phi_1 B)\ y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

for $ARIMA(1,1,1)$

$$(1 - \phi_1 B)\ z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} \qquad ;\ z_t = \nabla y_t = y_t - y_{t-1} = (1 - B) y_t$$

$$z_t - \phi_1 B\ z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$z_t - \phi_1\ z_{t-1} = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$z_t = \phi_1\ z_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - y_{t-1} = \phi_1 (y_{t-1} - y_{t-2}) + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - y_{t-1} = \phi_1 y_{t-1} - \phi_1 y_{t-2} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t = y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$\boxed{y_t = (1 + \phi_1)\ y_{t-1} - \phi_1 y_{t-2} + \varepsilon_t - \theta_1 \varepsilon_{t-1}}$$