

Chapter 3: Random Time Series Model

The random process $\{y_t\}$ is called **general liner process** if it's possible to express it in the form:

$$y_t = \mu + \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \dots, \quad t = 0, \pm 1, \pm 2, \dots$$

where, $\varepsilon_t \sim \text{iid } WN(0, \sigma_\varepsilon^2)$, μ is a constant and $\{\psi_t\}$ is a sequence of fixed values.

the process $\{y_t\}$ is stationary if one of the following conditions is satisfied:

1. ψ_i are finite.
2. $\sum_{i=0}^{\infty} \psi_i^2 < \infty$.

Question 1:

For the model $y_t = \varepsilon_t + 0.5y_{t-1}$ find π -weights and ψ -weights.

The π -weights formula $y_t = \varepsilon_t + \pi_1 y_{t-1} + \pi_2 y_{t-2} + \pi_3 y_{t-3} + \dots$

$$\pi_1 = 0.5 \quad \pi_2 = 0 \quad \pi_3 = 0$$

The ψ -weights formula $y_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \psi_3 \varepsilon_{t-3} + \dots$

And for ψ weight, we find:

$$y_t = \varepsilon_t + 0.5y_{t-1}$$

$$y_{t-1} = \varepsilon_{t-1} + 0.5y_{t-2}$$

$$y_{t-2} = \varepsilon_{t-2} + 0.5y_{t-3}$$

$$y_{t-3} = \varepsilon_{t-3} + 0.5y_{t-4}$$

$$y_t = \varepsilon_t + 0.5[y_{t-1}]$$

$$y_t = \varepsilon_t + 0.5[\varepsilon_{t-1} + 0.5y_{t-2}]$$

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2[y_{t-2}]$$

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2[\varepsilon_{t-2} + 0.5y_{t-3}]$$

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2\varepsilon_{t-2} + 0.5^3 y_{t-3}$$

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2\varepsilon_{t-2} + 0.5^3[y_{t-3}]$$

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2\varepsilon_{t-2} + 0.5^3[\varepsilon_{t-3} + 0.5y_{t-4}]$$

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2\varepsilon_{t-2} + 0.5^3\varepsilon_{t-3} + 0.5^4 y_{t-4}$$

$$\psi_1 = 0.5 \quad \psi_2 = 0.5^2 = 0.25 \quad \psi_3 = 0.5^3 = 0.125$$

Question 2:

For the model $y_t = \varepsilon_t - 0.3\varepsilon_{t-1}$ find the first three π weights and first three ψ weights.

solution:

The ψ weights formula $y_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \dots$

$$\psi_1 = -0.3 \quad \psi_2 = 0 \quad \psi_3 = 0$$

The π weights formula $y_t = \varepsilon_t + \pi_1y_{t-1} + \pi_2y_{t-2} + \pi_3y_{t-3} + \dots$

And for π weight, we find:

$$y_t = \varepsilon_t - 0.3\varepsilon_{t-1} \Rightarrow \varepsilon_t = y_t + 0.3\varepsilon_{t-1}$$

$$\varepsilon_{t-1} = y_{t-1} + 0.3\varepsilon_{t-2}$$

$$\varepsilon_{t-2} = y_{t-2} + 0.3\varepsilon_{t-3}$$

$$\varepsilon_{t-3} = y_{t-3} + 0.3\varepsilon_{t-4}$$

$$\varepsilon_t = y_t + 0.3[\varepsilon_{t-1}]$$

$$\varepsilon_t = y_t + 0.3[y_{t-1} + 0.3\varepsilon_{t-2}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2\varepsilon_{t-2}$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2[\varepsilon_{t-2}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2[y_{t-2} + 0.3\varepsilon_{t-3}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2y_{t-2} + 0.3^3\varepsilon_{t-3}$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2y_{t-2} + 0.3^3[\varepsilon_{t-3}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2y_{t-2} + 0.3^3[y_{t-3} + 0.3\varepsilon_{t-4}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2y_{t-2} + 0.3^3y_{t-3} + 0.3^4\varepsilon_{t-4}$$

$$y_t = \varepsilon_t - 0.3y_{t-1} - 0.3^2y_{t-2} - 0.3^3y_{t-3} - 0.3^4\varepsilon_{t-4}$$

$$\pi_1 = -0.3 \quad \pi_2 = -0.3^2 \quad \pi_3 = -0.3^3$$

- **Finding π -weights and ψ -weights for mixed:**

The ψ - weights formula

$$y_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \psi_3 \varepsilon_{t-3} + \dots, \quad t = 0, \pm 1, \pm 2, \dots$$

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots$$

The π - weights formula

$$y_t = \varepsilon_t + \pi_1 y_{t-1} + \pi_2 y_{t-2} + \pi_3 y_{t-3} + \dots, \quad t = 0, \pm 1, \pm 2, \dots$$

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \dots$$

To find ψ - weights $\phi(B) \psi(B) = \theta(B)$

To find π - weights $\theta(B) \pi(B) = \phi(B)$

Question 3:

For the model $y_t = 0.7y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$

find ψ -weights and π -weights.

solution:

to find the ψ -weight and π -weights:

$$y_t = 0.7y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$$

$$y_t - 0.7y_{t-1} = \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$$

$$y_t - 0.7y_{t-1} = \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$$

$$y_t - 0.7By_t = \varepsilon_t + 0.6 B\varepsilon_t - 0.4 B^2\varepsilon_t$$

$$(1 - 0.7B)y_t = (1 + 0.6 B - 0.4 B^2)\varepsilon_t$$

$$\phi(B) y_t = \theta(B) \varepsilon_t$$

To find ψ - weights $\boxed{\phi(B) \psi(B) = \theta(B)}$

equating the coefficients of B^j in both sides, we have:

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^0: \boxed{\psi_0 = 1}$$

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^1: \psi_1 - 0.7\psi_0 = 0.6 \Rightarrow \boxed{\psi_1 = 1.3}$$

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^2: \psi_2 - 0.7\psi_1 = -0.4 \Rightarrow \boxed{\psi_2 = 0.3}$$

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^3: \psi_3 - 0.7\psi_2 = 0 \Rightarrow \boxed{\psi_3 = 0.21}$$

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^4: \psi_4 - 0.7\psi_3 = 0$$

$$B^5: \psi_5 - 0.7\psi_4 = 0$$

$$\psi\text{- weights} = \begin{cases} \psi_0 = 1 \\ \psi_1 = 1.3 \\ \psi_2 = 0.3 \\ \psi_j - 0.7\psi_{j-1} = 0 \quad ; \quad j \geq 3 \end{cases}$$

To find π -weights $\theta(B) \pi(B) = \phi(B)$

$$(1 + 0.6B - 0.4B^2)(\pi_0 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \dots) = (1 - 0.7B)$$

$$B^0: \pi_0 = 1$$

$$B^1: \pi_1 + 0.6\pi_0 = -0.7 \Rightarrow \pi_1 = -1.3$$

$$B^2: \pi_2 + 0.6\pi_1 - 0.4\pi_0 = 0$$

$$B^3: \pi_3 + 0.6\pi_2 - 0.4\pi_1 = 0$$

$$\pi\text{-weights} = \begin{cases} \pi_0 = 1 \\ \pi_1 = -1.3 \\ \pi_i + 0.6\pi_{i-1} - 0.4\pi_{i-2} = 0 & ; i \geq 2 \end{cases}$$

Question 4:

For the model $y_t = 0.1y_{t-1} + \varepsilon_t + 0.2\varepsilon_{t-1}$

find ψ -weights and π -weights.

to find the ψ -weight and π -weights:

$$y_t = 0.1y_{t-1} + \varepsilon_t + 0.2\varepsilon_{t-1}$$

$$y_t - 0.1y_{t-1} = \varepsilon_t + 0.2\varepsilon_{t-1}$$

$$(1 - 0.1B)y_t = (1 + 0.2B)\varepsilon_t$$

$$\phi(B) y_t = \theta(B) \varepsilon_t$$

- To find ψ - weights $\phi(B) \psi(B) = \theta(B)$

$$(1 - 0.1B) (\psi_0 + \psi_1 B + \psi_2 B^2 + \dots) = (1 + 0.2B)$$

$$B^0: \psi_0 = 1$$

$$B^1: \psi_1 - 0.1\psi_0 = 0.2 \Rightarrow \psi_1 = 0.3$$

$$B^2: \psi_2 - 0.1\psi_1 = 0$$

$$\psi\text{- weights} = \begin{cases} \psi_0 = 1 \\ \psi_1 = 0.3 \\ \psi_j - 0.1\psi_{j-1} = 0 \quad ; \quad j \geq 2 \end{cases}$$

- To find π - weights $\theta(B) \pi(B) = \phi(B)$

$$(1 + 0.2B) (\pi_0 + \pi_1 B + \pi_2 B^2 + \dots) = (1 - 0.1B)$$

$$B^0: \pi_0 = 1$$

$$B^1: \pi_1 + 0.2\pi_0 = -0.1 \Rightarrow \pi_1 = -0.3$$

$$B^2: \pi_2 + 0.2\pi_1 = 0$$

$$\pi\text{- weights} = \begin{cases} \pi_0 = 1 \\ \pi_1 = -0.3 \\ \pi_i + 0.2\pi_{i-1} = 0 \quad ; \quad i \geq 1 \end{cases}$$

- *Auto-regressive processes:*

$$AR(p): y_t = \varepsilon_t + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p}$$

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \dots + \phi_p \rho(k-p) ; k = 1, 2, \dots$$

$$AR(1): y_t = \varepsilon_t + \phi_1 y_{t-1}$$

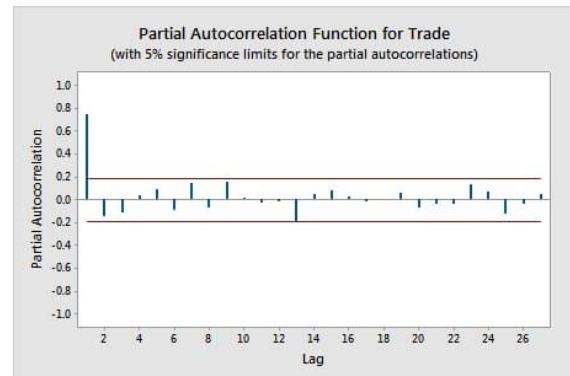
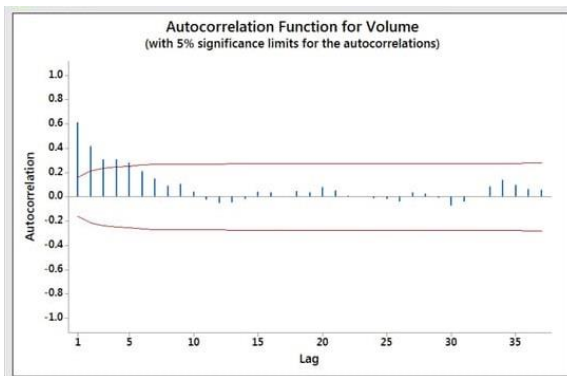
AR models are always invertible.

Stationary condition for AR (1):

$$|\phi_1| < 1 \quad \text{or} \quad |B| > 1$$

Auto-correlation function (ACF) for AR (1):

$$\rho(k) = \phi^k ; k = 1, 2, \dots$$



Question 5:

Assume the process $\{y_t\}$ follow the AR (1) model with parameter $\phi_1 = 0.4$, and assuming that $E(y_t) = 0$

1. write down the mathematical form of this model.

$$y_t = \varepsilon_t + 0.4y_{t-1},$$

where $\varepsilon_t \sim \text{iid } N(0, \sigma_\varepsilon^2)$ is the white noise process

2. is $\{y_t\}$ stationary, why?

the stationary condition for the AR (1) is:

$$|\phi_1| < 1 \Rightarrow |0.4| < 1$$

so, this process is stationary.

3. Drive the autocorrelation function ρ_k

$$y_t = \varepsilon_t + 0.4y_{t-1},$$

multiply both sides by y_{t-k} and taking the expectation.

$$y_t y_{t-k} = \varepsilon_t y_{t-k} + 0.4 y_{t-1} y_{t-k}$$

$$E(y_t y_{t-k}) = \underbrace{E(\varepsilon_t y_{t-k})}_0 + 0.4 E(y_{t-1} y_{t-k})$$

$$\gamma_k = 0.4 \gamma_{k-1} \quad \text{for } k \geq 1$$

dividing both sides by γ_0

$$\frac{\gamma_k}{\gamma_0} = 0.4 \frac{\gamma_{k-1}}{\gamma_0}$$

$$\rho_k = 0.4 \rho_{k-1} \quad \text{for } k \geq 1$$

$$\rho_1 = 0.4 \rho_0 = 0.4(1) = 0.4$$

$$\rho_2 = 0.4 \rho_1 = 0.4(0.4) = 0.4^2$$

$$\rho_3 = 0.4 \rho_2 = 0.4(0.4^2) = 0.4^3$$

$$\rho_k = 0.4 \rho_{k-1} = 0.4(0.4^{k-1}) = 0.4^k$$

$$AR(2): y_t = \varepsilon_t + \phi_1 y_{t-1} + \phi_2 y_{t-2}$$

AR models are always invertible.

Stationary condition for AR (2):

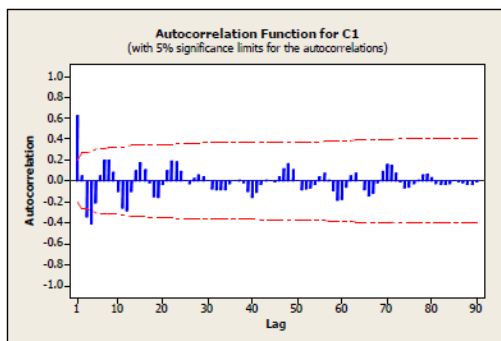
$$|\phi_2| < 1$$

$$\phi_2 + \phi_1 < 1$$

$$\phi_2 - \phi_1 < 1$$

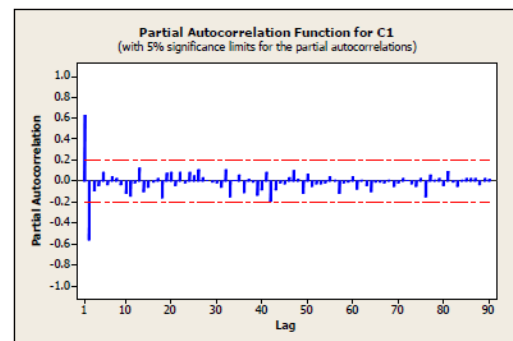
Auto-correlation function (ACF) for AR (2):

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2); \quad k = 1, 2, \dots$$



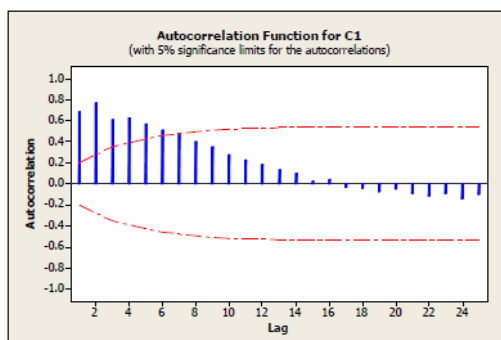
ACF for AR(2) model when

$$\phi_1 = 1, \phi_2 = -0.5$$



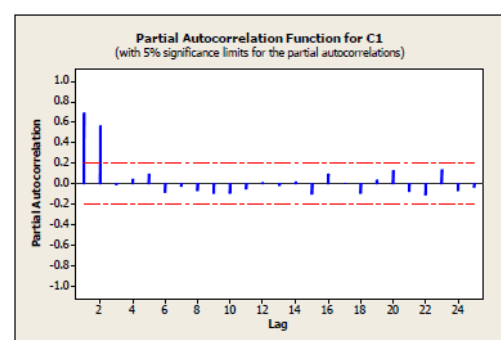
PACF for AR(2) model when

$$\phi_1 = 1, \phi_2 = -0.5$$



ACF for AR(2) model when

$$\phi_1 = 0.4, \phi_2 = 0.5$$



PACF for AR(2) model when

$$\phi_1 = 0.4, \phi_2 = 0.5$$

Question 6:

In the following question assume $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

For the following model, state its type, and check its stationarity and/or invertibility:

$$y_t = 0.7y_{t-1} + 0.5y_{t-2} + \varepsilon_t$$

model type: *AR (2) or ARMA (2,0) or ARIMA (2,0,0)*

stationarity: *since $|\phi_2| = |0.5| < 1$*

$$\phi_2 - \phi_1 = 0.5 - 0.7 = -0.2 < 1$$

$$\phi_2 + \phi_1 = 0.5 + 0.7 = 1.2 > 1$$

then this model is not stationary

invertibility: *AR models are always invertible*

- Moving average:

$$MA(q) \quad y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

$$\rho(k) = \begin{cases} \frac{-\theta_k + \theta_1 \theta_{k+1} + \dots + \theta_{q-k} \theta_q}{(1 + \theta_1^2 + \theta_2^2 + \dots + \theta_q^2)} & ; \quad k = 1, 2, \dots, q \\ 0 & ; \quad k > q \end{cases}$$

$$MA(1): \quad y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

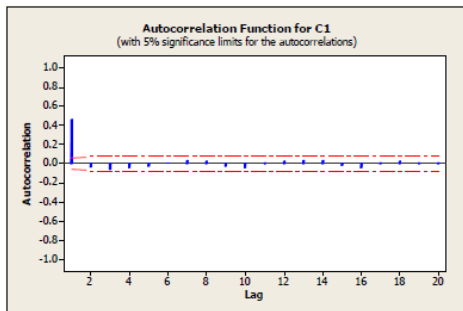
MA models are always stationary.

Invertibility conditions for the MA (1):

$$|\theta_1| < 1$$

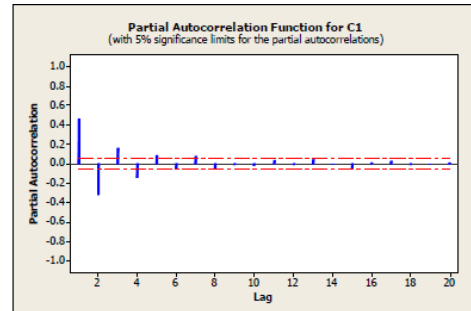
Auto-correlation function (ACF) for MA (1):

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\frac{\theta}{(1 + \theta^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$



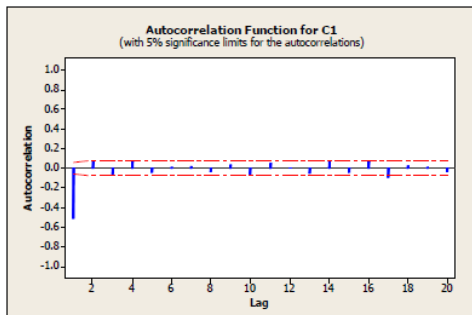
ACF for MA(1) model when

$$\theta_1 = -0.7$$



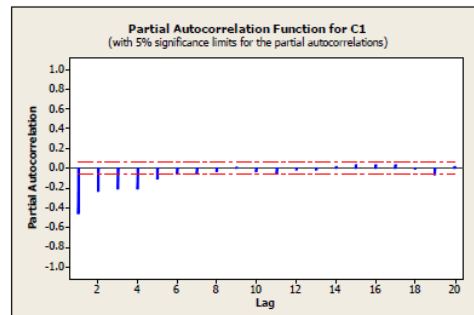
PACF for MA(1) model when

$$\theta_1 = -0.7$$



ACF for MA(1) model when

$$\theta_1 = 0.7$$



PACF for MA(1) model when

$$\theta_1 = 0.7$$

$$MA(2): y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

MA models are always stationary.

Invertibility conditions for the MA (2):

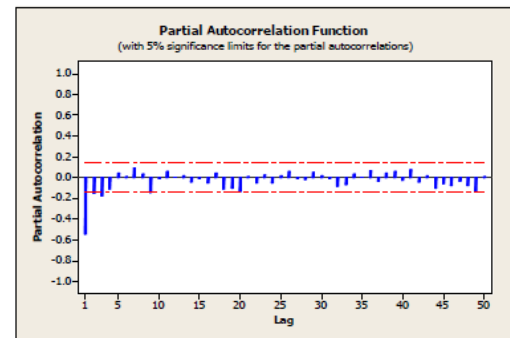
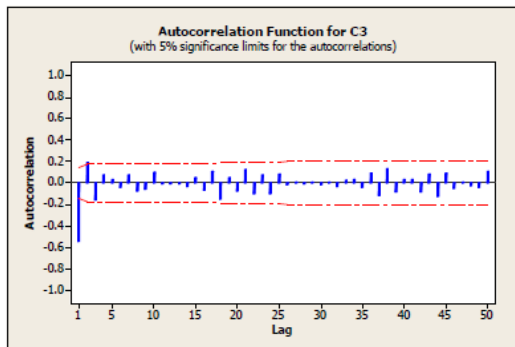
$$|\theta_2| < 1$$

$$\theta_2 + \theta_1 < 1$$

$$\theta_2 - \theta_1 < 1$$

Auto-correlation function (ACF) for MA (2):

$$\rho(k) = \begin{cases} \frac{-\theta_1(1-\theta_2)}{(1+\theta_1^2+\theta_2^2)} & k = 1 \\ \frac{-\theta_2}{(1+\theta_1^2+\theta_2^2)} & k = 2 \\ 0 & k \geq 2 \end{cases}$$

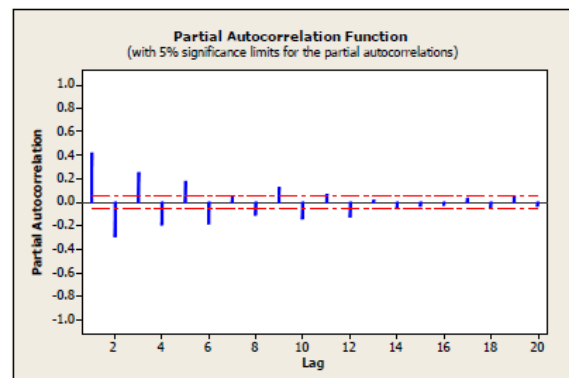
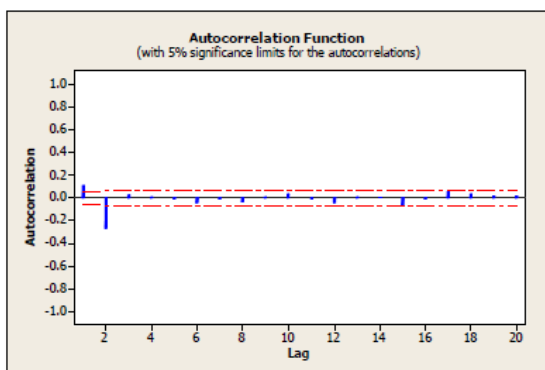


ACF for MA(2) model when

$$\theta_1 = 0.7, \theta_2 = -0.1$$

PACF for MA(2) model when

$$\theta_1 = 0.7, \theta_2 = -0.1$$



ACF for MA(2) model when

$$\theta_1 = 1, \theta_2 = -0.7$$

PACF for MA(2) model when

$$\theta_1 = 1, \theta_2 = -0.7$$

Question 7:

For MA (1) process with $\rho(1) = 0.4$,

Find the value of θ that make this model is invertible.

$$y_t = \varepsilon_t - \theta \varepsilon_{t-1}$$

$$\text{Var}(y_t) = \text{Var}(\varepsilon_t - \theta \varepsilon_{t-1}) = \sigma_\varepsilon^2 + \theta^2 \sigma_\varepsilon^2 = (1 + \theta^2) \sigma_\varepsilon^2$$

$$\text{Cov}(y_t, y_{t-k}) = E[(\varepsilon_t - \theta \varepsilon_{t-1})(\varepsilon_{t-k} - \theta \varepsilon_{t-k-1})]$$

$$\text{for } k = 1 \quad E[(\varepsilon_t - \theta \varepsilon_{t-1})(\varepsilon_{t-1} - \theta \varepsilon_{t-2})] = -\theta \sigma_\varepsilon^2$$

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\frac{\theta}{(1+\theta^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

$$\rho(1) = \frac{\gamma_1}{\gamma_0} = \frac{-\theta \sigma_\varepsilon^2}{(1+\theta^2)\sigma_\varepsilon^2} = \frac{-\theta}{1+\theta^2} = 0.4$$

$$\frac{-\theta}{1+\theta^2} = 0.4 \Rightarrow -\theta = 0.4 + 0.4\theta^2$$

$$\Rightarrow 0.4\theta^2 + \theta + 0.4 = 0$$

or we can use the calculator

$$\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(0.4)(0.4)}}{2(0.4)} = (-0.5) \text{ or } (-2)$$

MA (1) is invertibility $\Rightarrow |\theta| < 1$ then, $\theta = -0.5$

Question 8:

Assume the process $\{y_t\}$ follow the MA (2) model with parameters

$$\theta_1 = 0.4 \text{ and } \theta_2 = -0.8$$

1. write down the mathematical form of this model.

$$y_t = \varepsilon_t - 0.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

where ε_t is $WN(0, \sigma_\varepsilon^2)$

2. is $\{y_t\}$ invertible?

the invertible condition for the MA (2) is:

- $|\theta_2| < 1 \Rightarrow |-0.8| < 1$
- $\theta_2 + \theta_1 < 1 \Rightarrow -0.8 + 0.4 \Rightarrow -0.4 < 1$
- $\theta_2 - \theta_1 < 1 \Rightarrow -0.8 - 0.4 \Rightarrow -1.2 < 1$

the invertibility conditions are satisfied.

Question 9:

In the following question assume $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

For the following model, **state its type**, and check its **stationarity and/or invertibility**:

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} - 1.2\varepsilon_{t-2}$$

model type: **MA (2) or ARMA (0,2) or ARIMA (0,0,2)**

Stationarity: **Moving Average models are always stationary**

Invertibility: **since $\theta_2 = 1.2 > 1$, then this model is not invertible.**

Autoregressive-Moving average processes:

$$ARMA(p, q) \quad y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

for ARMA(p, q):

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \dots - \phi_p y_{t-p} = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

$$y_t - \phi_1 B y_t - \phi_2 B^2 y_t - \dots - \phi_p B^p y_t = \varepsilon_t - \theta_1 B \varepsilon_t - \theta_2 B^2 \varepsilon_t - \dots - \theta_q B^q \varepsilon_t$$

$$(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p) y_t = (1 - \theta_1 B - \theta_2 B^2 - \dots - \theta_q B^q) \varepsilon_t$$

$$\phi(B) y_t = \theta(B) \varepsilon_t$$

then, ARIMA(p, d, q) is:

$$\phi(B) \nabla^d y_t = \theta(B) \varepsilon_t$$

$$\Rightarrow \phi(B) z_t = \theta(B) \varepsilon_t$$

Model	ARMA(p, q)	ARIMA(p, d, q)
AR(p)	ARMA(p, 0)	ARIMA(p, 0, 0)
AR(2)	ARMA(2, 0)	ARIMA(2, 0, 0)
MA(q)	ARMA(0, q)	ARIMA(0, 0, q)
MA(1)	ARMA(0, 1)	ARIMA(0, 0, 1)

$$\text{ARMA}(1,1) \quad y_t = \phi_1 y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

Auto-correlation function (ACF) for ARMA (1,1):

$$\rho_k = \begin{cases} \frac{(\phi_1 - \theta_1)(1 - \phi_1 \theta_1)}{(1 + \theta_1^2 - 2\phi_1 \theta_1)} & ; k = 1 \\ \phi_1 \rho_{k-1} & ; k = 2, 3, \dots \end{cases}$$

Question 10:

In the following question assume $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

For the following model, **state its type**, and check its **stationarity and/or invertibility**:

$$y_t = 1.5y_{t-1} + \varepsilon_t + 0.25\varepsilon_{t-1}$$

model type: **ARMA (1,1) or ARIMA (1,0,1)**

stationarity: **since $|\phi_1| = |1.5| > 1$, then this model is not stationary**

invertibility: **since $|\theta_1| = |-0.25| < 1$, then this model is invertible.**

Question 11:

for the following model $y_t = 0.2 y_{t-1} + \varepsilon_t - 1.1 \varepsilon_{t-1} + 0.18 \varepsilon_{t-2}$

$\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$, state its type, and check its stationary and / or invertibility.

$$y_t - 0.2 y_{t-1} = \varepsilon_t - 1.1 \varepsilon_{t-1} + 0.18 \varepsilon_{t-2}$$

$$y_t - 0.2B y_t = \varepsilon_t - 1.1 B \varepsilon_t + 0.18 B^2 \varepsilon_t$$

$$(1 - 0.2B) y_t = (1 - 1.1 B + 0.18 B^2) \varepsilon_t$$

$$(1 - 0.2B) y_t = (1 - 0.2 B)(1 - 0.9 B) \varepsilon_t$$

$$y_t = (1 - 0.9 B) \varepsilon_t$$

model type: **MA (1) or ARMA (0,1) or ARIMA (0,0,1)**

stationarity: **MA (1) always stationary**

invertibility: **since $|\theta_1| = |0.9| < 1$ then this model is invertible.**

Question 12:

write the model ARIMA(1,1,1) in the final form in terms of

$$z_t = \nabla y_t, y_t \text{ and } \varepsilon_t$$

for ARMA(1,1)

$$y_t = \phi_1 y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - \phi_1 y_{t-1} = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - \phi_1 B y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$(1 - \phi_1 B) y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$


for ARIMA(1,1,1)

$$(1 - \phi_1 B) z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} \quad ; \quad z_t = \nabla y_t = y_t - y_{t-1} = (1 - B)y_t$$

$$z_t - \phi_1 B z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$z_t - \phi_1 z_{t-1} = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$z_t = \phi_1 z_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$


$$y_t - y_{t-1} = \phi_1 (y_{t-1} - y_{t-2}) + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - y_{t-1} = \phi_1 y_{t-1} - \phi_1 y_{t-2} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t = y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$\boxed{y_t = (1 + \phi_1) y_{t-1} - \phi_1 y_{t-2} + \varepsilon_t - \theta_1 \varepsilon_{t-1}}$$