Chapter 4: Parameter Estimation

- *Method of moments:*

$AR(1): y_t = \varepsilon_t + \phi_1 y_{t-1}$

$\hat{\phi}_1 = \hat{\rho}_1 = r_1$

$AR(2): y_t = \varepsilon_t + \phi_1 y_{t-1} + \phi_2 y_{t-2}$

$\hat{\phi}_1 = \dfrac{r_1(1-r_2)}{1-r_1^2} \qquad \hat{\phi}_2 = \dfrac{r_2-r_1^2}{1-r_1^2}$

$MA(1): y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$

$r_1 = \dfrac{-\theta_1}{1+\theta_1^2}$

$r_1\,\theta^2 + \theta + r_1 = 0$

$\hat{\theta}_1 = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-1 \pm \sqrt{1-4r_1^2}}{2r_1}$

ACF of MA (1)

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\dfrac{\theta_1}{(1+\theta_1^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Estimation the σ_α^2

for AR(p): $\hat{\sigma}_\varepsilon^2 = \left(1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2 - \cdots - \hat{\phi}_p r_p\right) S^2$

for AR(1): $\hat{\sigma}_\varepsilon^2 = (1 - r_1^2)S^2 \;\; ; \;\; \hat{\phi}_1 = r_1$

for AR(2): $\hat{\sigma}_\varepsilon^2 = \left(1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2\right) S^2 \;\; ; \;\; \hat{\phi}_1 = \dfrac{r_1(1-r_2)}{1-r_1^2} \;\; and \;\; \hat{\phi}_2 = \dfrac{r_2-r_1^2}{1-r_1^2}$

for MA(q): $\hat{\sigma}_\varepsilon^2 = \dfrac{S^2}{\left(1+\hat{\theta}_1^2+\hat{\theta}_2^2+\cdots+\hat{\theta}_q^2\right)}$

for MA(1): $\hat{\sigma}_\varepsilon^2 = \dfrac{S^2}{\left(1+\hat{\theta}_1^2\right)}$

for ARMA(1,1): $\hat{\sigma}_\varepsilon^2 = \dfrac{1-\hat{\phi}_1^2}{\left(1-2\,\hat{\theta}_1\hat{\phi}_1+\hat{\theta}_1^2\right)}S^2$

- *Least squares methods:*

for AR(1):

$$(y_t - \mu) = \varepsilon_t + \phi_1(y_{t-1} - \mu)$$

$$\varepsilon_t = (y_t - \mu) - \phi_1(y_{t-1} - \mu)$$

$$S(\phi_1, \mu) = \sum_{t=2}^{n} \varepsilon_t^2 = \sum_{t=2}^{n}[(y_t - \mu) - \phi_1(y_{t-1} - \mu)]^2$$

$$\frac{\partial S}{\partial \mu} = 2\sum_{t=2}^{n}[(y_t - \mu) - \phi_1(y_{t-1} - \mu)][-1 + \phi_1] = 0$$

$$\Rightarrow \sum_{t=2}^{n}[(y_t - \mu) - \phi_1(y_{t-1} - \mu)] = 0$$

$$\Rightarrow \sum_{t=2}^{n}[y_t - \mu - \phi_1 y_{t-1} + \phi_1\mu] = 0$$

$$\Rightarrow \sum_{t=2}^{n} y_t - \sum_{t=2}^{n} \mu - \phi_1 \sum_{t=2}^{n} y_{t-1} + \sum_{t=2}^{n} \phi_1\mu = 0$$

$$\Rightarrow \sum_{t=2}^{n} y_t - (n-1)\mu - \phi_1 \sum_{t=2}^{n} y_{t-1} + (n-1)\phi_1\mu = 0$$

$$\Rightarrow \sum_{t=2}^{n} y_t - \phi_1 \sum_{t=2}^{n} y_{t-1} - (n-1)\mu + (n-1)\phi_1\mu = 0$$

$$\Rightarrow \sum_{t=2}^{n} y_t - \phi_1 \sum_{t=2}^{n} y_{t-1} - (1-\phi_1)(n-1)\mu = 0$$

$$\Rightarrow \sum_{t=2}^{n} y_t - \phi_1 \sum_{t=2}^{n} y_{t-1} = (1-\phi_1)(n-1)\mu$$

$$\Rightarrow \boxed{\hat{\mu} = \frac{\sum_{t=2}^{n} y_t - \phi_1 \sum_{t=2}^{n} y_{t-1}}{(n-1)(1-\phi_1)}}$$

for large n

$$\sum_{t=2}^{n} \frac{y_t}{n-1} = \sum_{t=2}^{n} \frac{y_{t-1}}{n-1} \approx \bar{y}$$

$$\frac{\partial S}{\partial \phi_1} = 2\sum_{t=2}^{n}[(y_t - \bar{y}) - \phi_1(y_{t-1} - \bar{y})](y_{t-1} - \bar{y}) = 0$$

$$\Rightarrow \sum_{t=2}^{n}(y_t - \bar{y})(y_{t-1} - \bar{y}) - \phi_1 \sum_{t=2}^{n}(y_{t-1} - \bar{y})^2 = 0$$

$$\Rightarrow \boxed{\hat{\phi}_1 = \frac{\sum_{t=2}^{n}(y_t - \bar{y})(y_{t-1} - \bar{y})}{\sum_{t=2}^{n}(y_{t-1} - \bar{y})^2}}$$

MA(1) see page 95-97

Question 1:

Using the method of moments, answer the following questions:

a. What is the estimate of the variance of the process $\{y_t\}$, i.e. $\gamma(0)$ in the ARMA(p,q) models?

$\gamma(0)$ is estimated by the method of moments as $S^2 = \sum_{i=1}^{n} \frac{(y_i - \bar{y})^2}{n-1}$

b. For the model $y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$:

   i. obtain the estimate of the parameter θ_1.

$$r_1 = \frac{-\theta_1}{1+\theta_1^2}$$

$$r_1 \theta^2 + \theta + r_1 = 0$$

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\dfrac{\theta_1}{(1+\theta_1^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

$$\Rightarrow \hat{\theta}_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4r_1^2}}{2r_1}$$

   ii. obtain the estimate of white noise variance $\hat{\sigma}_\varepsilon^2$.

$$\gamma_0 = Var(y_t) = Var(\varepsilon_t - \theta_1 \varepsilon_{t-1}) = Var(\varepsilon_t) + Var(\theta_1 \varepsilon_{t-1})$$

$$\gamma_0 = \sigma_\varepsilon^2 + \theta_1^2 \sigma_\varepsilon^2$$

$$\gamma_0 = \sigma_\varepsilon^2 (1 + \theta_1^2)$$

$$\boxed{\hat{\sigma}_\varepsilon^2 = \frac{S^2}{(1+\hat{\theta}_1^2)}}$$

Question 2:

Using the method of moments, answer the following questions:

a. *For the model* $y_t = \phi_1 y_{t-1} + \varepsilon_t$:

We have the Yule-Walker equation

$$\rho_k = \phi_1 \, \rho_{k-1} \; ; \; k = 1,2,3, \dots$$

replace ρ_k *by* r_k *we got:*

$$r_k = \hat{\phi}_1 r_{k-1} \; ; \; k = 1,2,3, \dots$$

for $k = 1$: $\quad r_1 = \hat{\phi}_1 r_0 = \hat{\phi}_1$

$$\boxed{\hat{\phi}_1 = r_1}$$

b. *Obtain the estimate of white noise variance* $\hat{\sigma}_\varepsilon^2$

for AR (1), we have

$$\gamma_0 = Var(y_t) = Var(\phi_1 y_{t-1} + \varepsilon_t)$$

$$\gamma_0 = Var(\phi_1 y_{t-1}) + Var(\varepsilon_t)$$

$$\gamma_0 = \phi_1 \gamma_1 + \sigma_\varepsilon^2$$

$$\gamma_0 = \phi_1 \rho_1 \gamma_0 + \sigma_\varepsilon^2 \qquad where, \; \rho_1 = \frac{\gamma_1}{\gamma_0} \implies \boxed{\gamma_1 = \rho_1 \gamma_0}$$

$$\hat{\sigma}_\varepsilon^2 = \hat{\gamma}_0 - \hat{\phi}_1 r_1 \hat{\gamma}_0$$

$$\hat{\sigma}_\varepsilon^2 = \hat{\gamma}_0 \left(1 - \hat{\phi}_1 r_1\right)$$

$$\boxed{\hat{\sigma}_\varepsilon^2 = S^2 \left(1 - r_1^2\right)} \qquad where, \; \hat{\phi}_1 = r_1$$

Question 3:

Assume that 100 observations from an ARMA (1,1) model

$$Z_t - \phi_1 Z_{t-1} = \alpha_t - \theta_1 \alpha_{t-1}$$

Gave the following estimates $\hat{\sigma}_\varepsilon^2 = 10$, $\hat{\rho}_1 = 0.523$ and $\hat{\rho}_2 = 0.418$.

Find initial estimates for ϕ_1, θ_1 and σ_ε^2.

> *The ACF of ARMA (1,1) is*
>
> $$\rho_k = \begin{cases} \dfrac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{(1 + \theta_1^2 - 2\phi_1\theta_1)} & ; \ k = 1 \\ \phi_1 \rho_{k-1} & ; k = 2,3,\dots \end{cases}$$

- *Estimating ϕ_1:*

$$\rho_2 = \phi_1 \rho_1$$

$$r_2 = \hat{\phi}_1 r_1$$

$$\hat{\phi}_1 = \frac{r_2}{r_1} = \frac{0.418}{0.523} \implies \boxed{\hat{\phi}_1 = 0.799}$$

- *Estimating θ_1:*

$$r_1 = \frac{(\hat{\phi}_1 - \hat{\theta}_1)(1 - \hat{\phi}_1\hat{\theta}_1)}{1 + \hat{\theta}_1^2 - 2\hat{\phi}_1\hat{\theta}_1} = \frac{\hat{\phi}_1 - \hat{\phi}_1^2\hat{\theta}_1 - \hat{\theta}_1 + \hat{\phi}_1\hat{\theta}_1^2}{1 + \hat{\theta}_1^2 - 2\hat{\phi}_1\hat{\theta}_1}$$

$$0.523 = \frac{0.799 - 0.638\,\hat{\theta}_1 - \hat{\theta}_1 + 0.799\,\hat{\theta}_1^2}{1 + \hat{\theta}_1^2 - 1.598\,\hat{\theta}_1}$$

$$0.523 = \frac{0.799 - 1.638\,\hat{\theta}_1 + 0.799\,\hat{\theta}_1^2}{1 + \hat{\theta}_1^2 - 1.598\,\hat{\theta}_1}$$

$$0.523\left(1 + \hat{\theta}_1^2 - 1.598\,\hat{\theta}_1\right) = 0.799 - 1.638\,\hat{\theta}_1 + 0.799\,\hat{\theta}_1^2$$

$$0.523 + 0.523\,\hat{\theta}_1^2 - 0.836\,\hat{\theta}_1 = 0.799 - 1.638\,\hat{\theta}_1 + 0.799\,\hat{\theta}_1^2$$

$$0.799 - 1.638\,\hat{\theta}_1 + 0.799\,\hat{\theta}_1^2 - \left(0.523 + 0.523\,\hat{\theta}_1^2 - 0.836\,\hat{\theta}_1\right) = 0$$

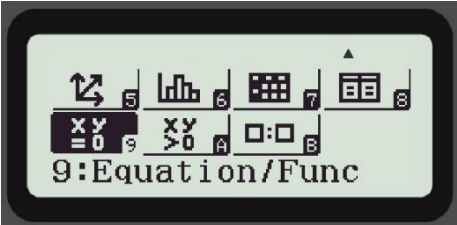$$0.276\,\hat{\theta}_1^2 - 0.802\,\hat{\theta}_1 + 0.276 = 0$$

$$\hat{\theta}_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

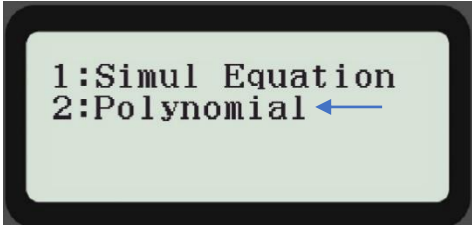$$\hat{\theta}_1 = \frac{-0.802 \pm \sqrt{0.802^2 - 4(0.276)(0.276)}}{2(0.276)}$$

$$\hat{\theta}_1 = 2.507 \quad or \quad \hat{\theta}_1 = 0.399$$

or we can use the calculator

1

2


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1:Simul Equation
2:Polynomial ←
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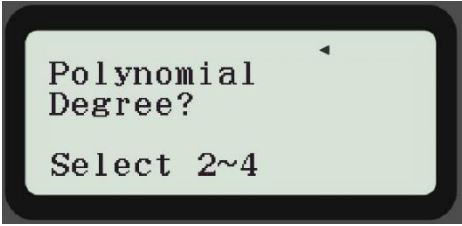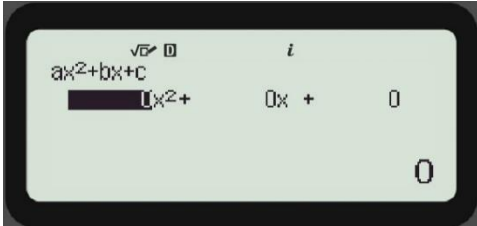
3


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Polynomial
Degree?

Select 2~4
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4

We will choose 0.399 to satisfy invertibility condition, which is $|\theta_1| < 1$.

- *Estimating σ_ε^2:*

$$\sigma_\varepsilon^2 = \frac{1 - \hat{\phi}_1^2}{1 - 2\hat{\theta}_1\hat{\phi}_1 + \theta_1^2} S^2$$

$$\sigma_\varepsilon^2 = \frac{1 - 0.799^2}{1 - 2(0.799)(0.399) + 0.399^2}\,(10) = 6.932$$

Question 4:

Assume that 100 observations from an AR(2) model

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \alpha_t$$

Gave the following sample ACF: $\hat{\rho}_1 = 0.8$, $\hat{\rho}_2 = 0.5$ and $\hat{\rho}_3 = 0.4$

Estimate ϕ_1 and ϕ_2.

The auto-correlation function for AR(2) is :

$$\rho_k = \phi_1(\rho_{k-1}) + \phi_2(\rho_{k-2}) \qquad ; k = 1, 2, \dots$$

$\hat{\rho}_k = r_k$

$r_1 = \hat{\phi}_1 r_0 + \hat{\phi}_2 r_1$
$r_1 = \hat{\phi}_1(1) + \hat{\phi}_2 r_1$

$\implies \hat{\phi}_1 = r_1 - \hat{\phi}_2 r_1 \quad \dots \quad (1)$

$r_2 = \hat{\phi}_1 r_1 + \hat{\phi}_2 r_0$
$r_2 = \hat{\phi}_1 r_1 + \hat{\phi}_2$

$\implies \hat{\phi}_2 = r_2 - \hat{\phi}_1 r_1 \quad \dots \quad (2)$

substitute (2) in (1):	substitute (1) in (2):
$\hat{\phi}_1 = r_1 - \hat{\phi}_2 r_1$	$\hat{\phi}_2 = r_2 - \hat{\phi}_1 r_1$
$\hat{\phi}_1 = r_1(1 - \hat{\phi}_2)$	$\hat{\phi}_2 = r_2 - (r_1 - \hat{\phi}_2 r_1) r_1$
$\hat{\phi}_1 = r_1 \left(1 - (r_2 - \hat{\phi}_1 r_1)\right)$	$\hat{\phi}_2 = r_2 - r_1^2 + \hat{\phi}_2 r_1^2$
$\hat{\phi}_1 = r_1(1 - r_2 + \hat{\phi}_1 r_1)$	$\hat{\phi}_2 - \hat{\phi}_2 r_1^2 = r_2 - r_1^2$
$\hat{\phi}_1 = r_1 - r_1 r_2 + \hat{\phi}_1 r_1^2$	$\hat{\phi}_2(1 - r_1^2) = r_2 - r_1^2$
$\hat{\phi}_1 - \hat{\phi}_1 r_1^2 = r_1 - r_1 r_2$	$\hat{\phi}_2 = \dfrac{r_2 - r_1^2}{(1 - r_1^2)}$
$\hat{\phi}_1(1 - r_1^2) = r_1(1 - r_2)$	$\hat{\phi}_2 = \dfrac{r_2 - r_1^2}{1 - r_1^2} = \dfrac{0.5 - 0.8^2}{1 - 0.8^2} = -0.3889$
$\hat{\phi}_1 = \dfrac{r_1(1 - r_2)}{1 - r_1^2}$	
$\hat{\phi}_1 = \dfrac{r_1(1 - r_2)}{1 - r_1^2} = \dfrac{0.8(1 - 0.5)}{1 - 0.8^2} = 1.11$	

7

Other solution:

$r_1 = \hat{\phi}_1 + \hat{\phi}_2 r_1$	$r_2 = \hat{\phi}_1 r_1 + \hat{\phi}_2$
$0.8 = \hat{\phi}_1 + (0.8)\hat{\phi}_2$	$0.5 = 0.8\hat{\phi}_1 + \hat{\phi}_2 \dots (2)$
$\hat{\phi}_1 = 0.8 - 0.8\hat{\phi}_2 \dots (1)$	

Replace $\hat{\phi}_1$ in (2) using (1):

$0.5 = 0.8\hat{\phi}_1 + \hat{\phi}_2$

$0.5 = 0.8\left(0.8 - 0.8\hat{\phi}_2\right) + \hat{\phi}_2$

$0.5 = 0.64 - 0.64\hat{\phi}_2 + \hat{\phi}_2$

$0.5 = 0.64 + 0.36\hat{\phi}_2$

$-0.14 = 0.36\hat{\phi}_2$

$\hat{\phi}_2 = -0.3889$

from (1)

$\hat{\phi}_1 = 0.8 - 0.8\hat{\phi}_2$

$\hat{\phi}_1 = 0.8 - 0.8(-0.3889)$

$\hat{\phi}_1 = 1.1111$