

## Chapter 4: Parameter Estimation

- Method of moments:

$$AR(1): y_t = \varepsilon_t + \phi_1 y_{t-1}$$

$$\hat{\phi}_1 = \hat{\rho}_1 = r_1$$

$$AR(2): y_t = \varepsilon_t + \phi_1 y_{t-1} + \phi_2 y_{t-2}$$

$$\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2} \quad \hat{\phi}_2 = \frac{r_2-r_1^2}{1-r_1^2}$$

$$MA(1): y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$r_1 = \frac{-\theta_1}{1+\theta_1^2}$$

$$r_1 \theta^2 + \theta + r_1 = 0$$

$$\hat{\theta}_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4r_1^2}}{2r_1}$$

ACF of MA (1)

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\frac{\theta_1}{(1+\theta_1^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Estimation the  $\sigma_\varepsilon^2$

$$\text{for } AR(p): \hat{\sigma}_\varepsilon^2 = (1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2 - \dots - \hat{\phi}_p r_p) S^2$$

$$\text{for } AR(1): \hat{\sigma}_\varepsilon^2 = (1 - r_1^2) S^2 \quad ; \quad \hat{\phi}_1 = r_1$$

$$\text{for } AR(2): \hat{\sigma}_\varepsilon^2 = (1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2) S^2 \quad ; \quad \hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2} \quad \text{and} \quad \hat{\phi}_2 = \frac{r_2-r_1^2}{1-r_1^2}$$

$$\text{for } MA(q): \hat{\sigma}_\varepsilon^2 = \frac{S^2}{(1 + \hat{\theta}_1^2 + \hat{\theta}_2^2 + \dots + \hat{\theta}_q^2)}$$

$$\text{for } MA(1): \hat{\sigma}_\varepsilon^2 = \frac{S^2}{(1 + \hat{\theta}_1^2)}$$

$$\text{for } ARMA(1,1): \hat{\sigma}_\varepsilon^2 = \frac{1 - \hat{\phi}_1^2}{(1 - 2\hat{\theta}_1\hat{\phi}_1 + \hat{\theta}_1^2)} S^2$$

- *Least squares methods:*

for AR(1):

$$(y_t - \mu) = \varepsilon_t + \phi_1(y_{t-1} - \mu)$$

$$\varepsilon_t = (y_t - \mu) - \phi_1(y_{t-1} - \mu)$$

$$S(\phi_1, \mu) = \sum_{t=2}^n \varepsilon_t^2 = \sum_{t=2}^n [(y_t - \mu) - \phi_1(y_{t-1} - \mu)]^2$$

$$\frac{\partial S}{\partial \mu} = 2 \sum_{t=2}^n [(y_t - \mu) - \phi_1(y_{t-1} - \mu)] [-1 + \phi_1] = 0$$

$$\Rightarrow \sum_{t=2}^n [(y_t - \mu) - \phi_1(y_{t-1} - \mu)] = 0$$

$$\Rightarrow \sum_{t=2}^n [y_t - \mu - \phi_1 y_{t-1} + \phi_1 \mu] = 0$$

$$\Rightarrow \sum_{t=2}^n y_t - \sum_{t=2}^n \mu - \phi_1 \sum_{t=2}^n y_{t-1} + \sum_{t=2}^n \phi_1 \mu = 0$$

$$\Rightarrow \sum_{t=2}^n y_t - (n-1)\mu - \phi_1 \sum_{t=2}^n y_{t-1} + (n-1)\phi_1 \mu = 0$$

$$\Rightarrow \sum_{t=2}^n y_t - \phi_1 \sum_{t=2}^n y_{t-1} - (n-1)\mu + (n-1)\phi_1 \mu = 0$$

$$\Rightarrow \sum_{t=2}^n y_t - \phi_1 \sum_{t=2}^n y_{t-1} - (1 - \phi_1)(n-1)\mu = 0$$

$$\Rightarrow \sum_{t=2}^n y_t - \phi_1 \sum_{t=2}^n y_{t-1} = (1 - \phi_1)(n-1)\mu$$

$$\Rightarrow \boxed{\hat{\mu} = \frac{\sum_{t=2}^n y_t - \phi_1 \sum_{t=2}^n y_{t-1}}{(n-1)(1-\phi_1)}}$$

for large  $n$

$$\sum_{t=2}^n \frac{y_t}{n-1} = \sum_{t=2}^n \frac{y_{t-1}}{n-1} \approx \bar{y}$$

$$\frac{\partial S}{\partial \phi_1} = 2 \sum_{t=2}^n [(y_t - \bar{y}) - \phi_1(y_{t-1} - \bar{y})](y_{t-1} - \bar{y}) = 0$$

$$\Rightarrow \sum_{t=2}^n (y_t - \bar{y})(y_{t-1} - \bar{y}) - \phi_1 \sum_{t=2}^n (y_{t-1} - \bar{y})^2 = 0$$

$$\Rightarrow \boxed{\hat{\phi}_1 = \frac{\sum_{t=2}^n (y_t - \bar{y})(y_{t-1} - \bar{y})}{\sum_{t=2}^n (y_{t-1} - \bar{y})^2}}$$

MA(1) see page 95-97

**Question 1:**

**Using the method of moments, answer the following questions:**

a. What is the estimate of the variance of the process  $\{y_t\}$ , i.e.  $\gamma(0)$  in the ARMA(p,q) models?

$\gamma(0)$  is estimated by the method of moments as  $S^2 = \sum_{i=1}^n \frac{(y_i - \bar{y})^2}{n-1}$

b. For the model  $y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$  :

i. obtain the estimate of the parameter  $\theta_1$ .

$$r_1 = \frac{-\theta_1}{1+\theta_1^2}$$

$$r_1 \theta^2 + \theta + r_1 = 0$$

$$\Rightarrow \hat{\theta}_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4r_1^2}}{2r_1}$$

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\frac{\theta_1}{(1 + \theta_1^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

ii. obtain the estimate of white noise variance  $\hat{\sigma}_\varepsilon^2$  .

$$\gamma_0 = \text{Var}(y_t) = \text{Var}(\varepsilon_t - \theta_1 \varepsilon_{t-1}) = \text{Var}(\varepsilon_t) + \text{Var}(\theta_1 \varepsilon_{t-1})$$

$$\gamma_0 = \sigma_\varepsilon^2 + \theta_1^2 \sigma_\varepsilon^2$$

$$\gamma_0 = \sigma_\varepsilon^2 (1 + \theta_1^2)$$

$$\hat{\sigma}_\varepsilon^2 = \frac{S^2}{(1 + \hat{\theta}_1^2)}$$

**Question 2: Using the method of moments, answer the following questions:**

a. For the model  $y_t = \phi_1 y_{t-1} + \varepsilon_t$  :

We have the Yule-Walker equation

$$\rho_k = \phi_1 \rho_{k-1} ; k = 1, 2, 3, \dots$$

replace  $\rho_k$  by  $r_k$  we got:

$$r_k = \hat{\phi}_1 r_{k-1} ; k = 1, 2, 3, \dots$$

for  $k = 1$  :  $r_1 = \hat{\phi}_1 r_0 = \hat{\phi}_1$

$$\boxed{\hat{\phi}_1 = r_1}$$

b. Obtain the estimate of white noise variance  $\hat{\sigma}_\varepsilon^2$  .  
for AR (1), we have

$$\gamma_0 = \text{Var}(y_t) = \text{Var}(\phi_1 y_{t-1} + \varepsilon_t)$$

$$\gamma_0 = \text{Var}(\phi_1 y_{t-1}) + \text{Var}(\varepsilon_t)$$

$$\gamma_0 = \phi_1 \gamma_1 + \sigma_\varepsilon^2$$

$$\gamma_0 = \phi_1 \rho_1 \gamma_0 + \sigma_\varepsilon^2 \quad \text{where, } \rho_1 = \frac{\gamma_1}{\gamma_0} \Rightarrow \boxed{\gamma_1 = \rho_1 \gamma_0}$$

$$\hat{\sigma}_\varepsilon^2 = \hat{\gamma}_0 - \hat{\phi}_1 r_1 \hat{\gamma}_0$$

$$\hat{\sigma}_\varepsilon^2 = \hat{\gamma}_0 (1 - \hat{\phi}_1 r_1)$$

$$\boxed{\hat{\sigma}_\varepsilon^2 = S^2 (1 - r_1^2)} \quad \text{where, } \hat{\phi}_1 = r_1$$

Note:  $\gamma_0 = \phi_1^2 \text{Var}(y_{t-1}) + \text{Var}(\varepsilon_t) \dots (1)$

where,  $\gamma_1 = \text{cov}(y_t, y_{t-1}) = \text{cov}(\phi_1 y_{t-1} + \varepsilon_t, y_{t-1})$

$$= \phi_1 \text{cov}(y_{t-1}, y_{t-1}) + \text{cov}(\varepsilon_t, y_{t-1})$$

$$\gamma_1 = \phi_1 \text{var}(y_{t-1}) + 0 \gg \gg \phi_1 \gamma_1 = \phi_1^2 \text{var}(y_{t-1}) \dots (*)$$

Or By use Yule walker we get:

$$E(y_t y_{t-1}) = \phi_1 E(y_{t-1} y_{t-1}) + E(y_{t-1} \varepsilon_t)$$

$$E(y_{t-k} \varepsilon_t) = \begin{cases} \sigma_\varepsilon^2 & \text{at } k = 0 \\ 0 & \text{at } k \geq 1 \end{cases}$$

$$\text{at } k = 0 \gg \gg \gamma_0 = \phi_1 \gamma_1 + \sigma_\varepsilon^2$$

**Question 3:**

Assume that 100 observations from an ARMA (1,1) model

$$Z_t - \phi_1 Z_{t-1} = \alpha_t - \theta_1 \alpha_{t-1}$$

Gave the following estimates  $S_\varepsilon^2 = 10$ ,  $\hat{\rho}_1 = 0.523$  and  $\hat{\rho}_2 = 0.418$ .

Find initial estimates for  $\phi_1$ ,  $\theta_1$  and  $\sigma_\varepsilon^2$ .

The ACF of ARMA (1,1) is

$$\rho_k = \begin{cases} \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{(1 + \theta_1^2 - 2\phi_1\theta_1)} & ; k = 1 \\ \phi_1\rho_{k-1} & ; k = 2, 3, \dots \end{cases}$$

- Estimating  $\phi_1$ :

$$\rho_2 = \phi_1\rho_1$$

$$r_2 = \hat{\phi}_1 r_1$$

$$\hat{\phi}_1 = \frac{r_2}{r_1} = \frac{0.418}{0.523} \Rightarrow \boxed{\hat{\phi}_1 = 0.799}$$

- Estimating  $\theta_1$ :

$$r_1 = \frac{(\hat{\phi}_1 - \hat{\theta}_1)(1 - \hat{\phi}_1\hat{\theta}_1)}{1 + \hat{\theta}_1^2 - 2\hat{\phi}_1\hat{\theta}_1} = \frac{\hat{\phi}_1 - \hat{\phi}_1^2\hat{\theta}_1 - \hat{\theta}_1 + \hat{\phi}_1\hat{\theta}_1^2}{1 + \hat{\theta}_1^2 - 2\hat{\phi}_1\hat{\theta}_1}$$

$$0.523 = \frac{0.799 - 0.638\hat{\theta}_1 - \hat{\theta}_1 + 0.799\hat{\theta}_1^2}{1 + \hat{\theta}_1^2 - 1.598\hat{\theta}_1}$$

$$0.523 = \frac{0.799 - 1.638\hat{\theta}_1 + 0.799\hat{\theta}_1^2}{1 + \hat{\theta}_1^2 - 1.598\hat{\theta}_1}$$

$$0.523(1 + \hat{\theta}_1^2 - 1.598 \hat{\theta}_1) = 0.799 - 1.638 \hat{\theta}_1 + 0.799 \hat{\theta}_1^2$$

$$0.523 + 0.523 \hat{\theta}_1^2 - 0.836 \hat{\theta}_1 = 0.799 - 1.638 \hat{\theta}_1 + 0.799 \hat{\theta}_1^2$$

$$0.799 - 1.638 \hat{\theta}_1 + 0.799 \hat{\theta}_1^2 - (0.523 + 0.523 \hat{\theta}_1^2 - 0.836 \hat{\theta}_1) = 0$$

$$0.276 \hat{\theta}_1^2 - 0.802 \hat{\theta}_1 + 0.276 = 0$$

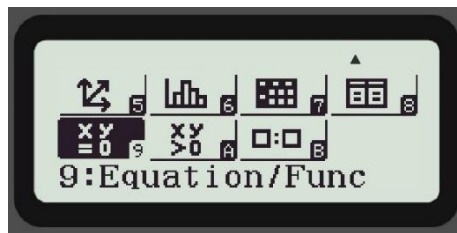
$$\hat{\theta}_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\hat{\theta}_1 = \frac{-0.802 \pm \sqrt{0.802^2 - 4(0.276)(0.276)}}{2(0.276)}$$

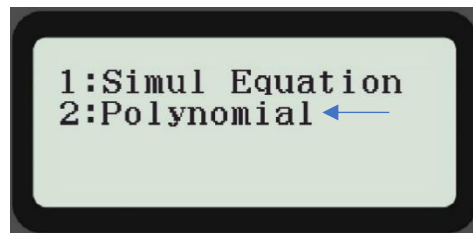
$$\hat{\theta}_1 = 2.507 \text{ or } \hat{\theta}_1 = 0.399$$

or we can use the calculator

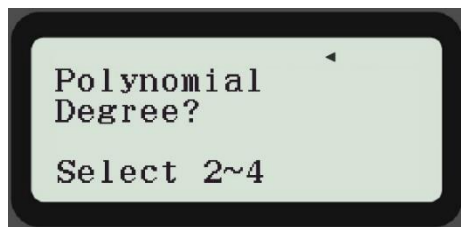
1



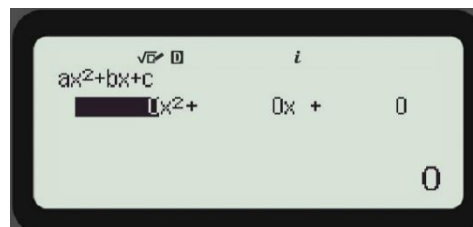
2



3



4



We will choose 0.399 to satisfy invertibility condition, which is  $|\theta_1| < 1$ .

- Estimating  $\sigma_\varepsilon^2$ :

$$\sigma_\varepsilon^2 = \frac{1 - \hat{\phi}_1^2}{1 - 2\hat{\theta}_1\hat{\phi}_1 + \theta_1^2} S^2$$

$$\sigma_\varepsilon^2 = \frac{1 - 0.799^2}{1 - 2(0.799)(0.399) + 0.399^2} (10) = 6.932$$

**Question 4:**

Assume that 100 observations from an AR(2) model

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \alpha_t$$

Gave the following sample ACF:  $\hat{\rho}_1 = 0.8$ ,  $\hat{\rho}_2 = 0.5$  and  $\hat{\rho}_3 = 0.4$

Estimate  $\phi_1$  and  $\phi_2$ .

The auto-correlation function for AR(2) is :

$$\rho_k = \phi_1(\rho_{k-1}) + \phi_2(\rho_{k-2}) \quad ; k = 1, 2, \dots$$

$$\hat{\rho}_k = r_k$$

$$r_1 = \hat{\phi}_1 r_0 + \hat{\phi}_2 r_1$$

$$r_1 = \hat{\phi}_1(1) + \hat{\phi}_2 r_1$$

$$\Rightarrow \hat{\phi}_1 = r_1 - \hat{\phi}_2 r_1 \quad \dots \quad (1)$$

$$r_2 = \hat{\phi}_1 r_1 + \hat{\phi}_2 r_0$$

$$r_2 = \hat{\phi}_1 r_1 + \hat{\phi}_2$$

$$\Rightarrow \hat{\phi}_2 = r_2 - \hat{\phi}_1 r_1 \quad \dots \quad (2)$$

substitute (2) in (1):	substitute (1) in (2):
$\hat{\phi}_1 = r_1 - \hat{\phi}_2 r_1$	$\hat{\phi}_2 = r_2 - \hat{\phi}_1 r_1$
$\hat{\phi}_1 = r_1(1 - \hat{\phi}_2)$	$\hat{\phi}_2 = r_2 - (r_1 - \hat{\phi}_2 r_1)r_1$
$\hat{\phi}_1 = r_1(1 - (r_2 - \hat{\phi}_1 r_1))$	$\hat{\phi}_2 = r_2 - r_1^2 + \hat{\phi}_2 r_1^2$
$\hat{\phi}_1 = r_1(1 - r_2 + \hat{\phi}_1 r_1)$	$\hat{\phi}_2 - \hat{\phi}_2 r_1^2 = r_2 - r_1^2$
$\hat{\phi}_1 = r_1 - r_1 r_2 + \hat{\phi}_1 r_1^2$	$\hat{\phi}_2(1 - r_1^2) = r_2 - r_1^2$
$\hat{\phi}_1 - \hat{\phi}_1 r_1^2 = r_1 - r_1 r_2$	$\hat{\phi}_2 = \frac{r_2 - r_1^2}{(1 - r_1^2)}$
$\hat{\phi}_1(1 - r_1^2) = r_1(1 - r_2)$	$\hat{\phi}_2 = \frac{r_2 - r_1^2}{1 - r_1^2} = \frac{0.5 - 0.8^2}{1 - 0.8^2} = -0.3889$
$\hat{\phi}_1 = \frac{r_1(1 - r_2)}{1 - r_1^2}$	

$\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2} = \frac{0.8(1-0.5)}{1-0.8^2} = 1.11$	
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*Other solution:*

$r_1 = \hat{\phi}_1 + \hat{\phi}_2 r_1$	$r_2 = \hat{\phi}_1 r_1 + \hat{\phi}_2$
$0.8 = \hat{\phi}_1 + (0.8) \hat{\phi}_2$	$0.5 = 0.8 \hat{\phi}_1 + \hat{\phi}_2 \dots (2)$
$\hat{\phi}_1 = 0.8 - 0.8 \hat{\phi}_2 \dots (1)$	

*Replace  $\hat{\phi}_1$  in (2) using (1):*

$$0.5 = 0.8 \hat{\phi}_1 + \hat{\phi}_2$$

$$0.5 = 0.8 (0.8 - 0.8 \hat{\phi}_2) + \hat{\phi}_2$$

$$0.5 = 0.64 - 0.64 \hat{\phi}_2 + \hat{\phi}_2$$

$$0.5 = 0.64 + 0.36 \hat{\phi}_2$$

$$-0.14 = 0.36 \hat{\phi}_2$$

$$\hat{\phi}_2 = -0.3889$$

*from (1)*

$$\hat{\phi}_1 = 0.8 - 0.8 \hat{\phi}_2$$

$$\hat{\phi}_1 = 0.8 - 0.8 (-0.3889)$$

$$\hat{\phi}_1 = 1.1111$$