Chapter 3

Fourier Series Representation

Introduction

Jean Baptiste Joseph **Fourier**

- Born in Auxerre, France
- Mathematician and physicist
- Developed Fourier series, Fourier transforms and their applications on heat and vibration
- Life span: 21 March 1768 – 16 May 1830
- Also known as an Egyptologist.

The response of LTI systems to complex exponentials

- For the study of LTI systems we represent signals as linear combinations of *basic signals* (unit impulse $\delta(t)$, complex exponential e^{st},…).

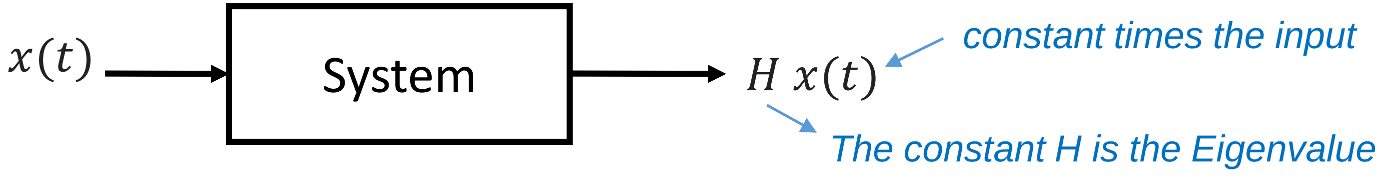
- The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude.

Continuous time: $e^{st} \rightarrow H(s)e^{st}$

Discrete time: $z^{n} \rightarrow H(z)z^{n}$

$H(s)$ and $H(z)$ are the amplitude factor (complex function of complex variable).

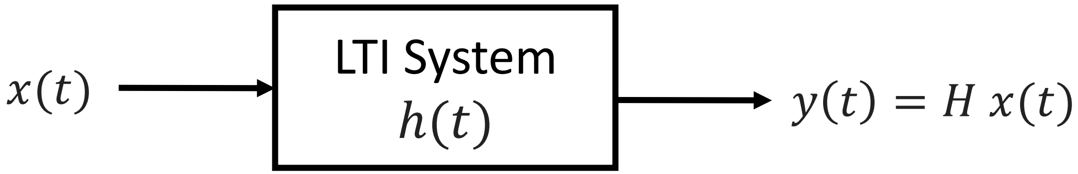
- A signal for which the system output is a (possibly complex) constant times the input is referred to as an *eigenfunction* of the system, and the amplitude factor is referred to as the system's eigenvalue.



Continuous time case

Complex exponentials are *eigenfunctions* of LTI systems

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$



IF: $x(t) = e^{st}$ (a complex exponential) ➡ $y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)} d\tau$ ⟵ *convolution*

➡ $y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-s\tau} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau$

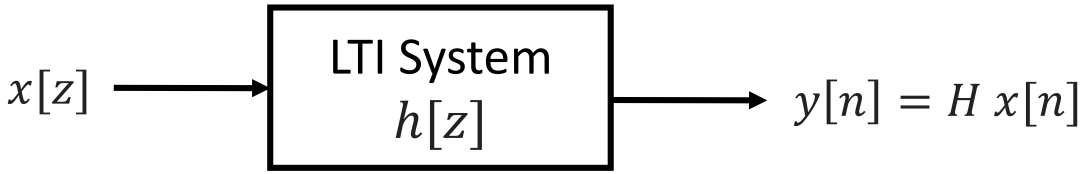
➡ $y(t) = H(s)\, e^{st}$ Where: $H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau$

The complex constant H(s) for a specific value of s is the '*eigenvalue*' associated with the *eigenfunction* e^{st}.

Discrete time case

Complex exponential sequences are eigenfunctions of discrete-time LTI systems.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$



IF: $x[n] = z^n$ (input the sequence) ⇨ $y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$

⇨ $$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,z^{n-k} = \sum_{k=-\infty}^{\infty} h[k]\,z^n\,z^{-k} = z^n \sum_{k=-\infty}^{\infty} h[k]\,z^{-k}$$

⇨ $y[n] = H[z]\,z^n$ With $H[z] = \sum_{k=-\infty}^{\infty} h[k]\,z^{-k}$

- The complex exponentials are '*eigenfunctions*' of LTI systems.
- The constant H(z) for a specific value of z is the '*eigenvalue*' associated with the *eigenfunction* z^n.

Fourier Series Representation of Continuous Time Periodic Signals

1. Linear combination of harmonically related complex exponentials

A signal is periodic, if, for some positive value of *T*, $x(t) = x(t+T), \quad \text{for all } t$ (1)

The *fundamental period* of x(t) is the minimum, positive, nonzero value of *T* for which equation (1) is satisfied.

Basic periodic signals:

- Sinusoidal: $x(t) = \cos(\omega_0 t)$
- Complex exponential: $x(t) = e^{j\omega_0 t}$

fundamental frequency: $\omega_0 = \frac{2\pi}{T}$

Harmonically related signals with the complex exponential:

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Fourier series representation of a periodic signal $x(t)$ with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

The term for *k* = 0 is a constant.

The terms for $k = \pm 1$ are the '*first harmonic components' or 'fundamental components*'.

The terms for $k = \pm 2$ are the '*second harmonic components'*.

The terms for $k = \pm N$ are the 'N^{th} harmonic components'.

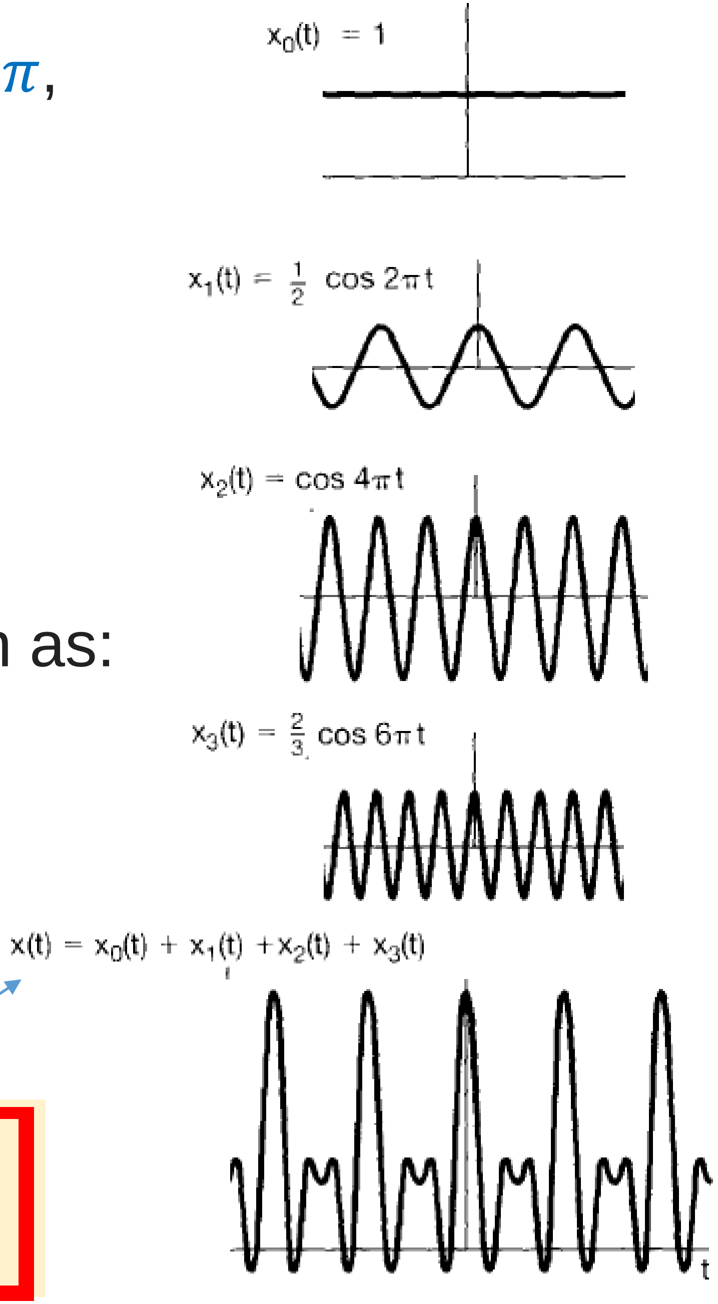
Example

Consider a periodic signal $x(t)$ with fundamental frequency 2π, expressed as:

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$$

where,

$$a_0 = 1$$
$$a_1 = a_{-1} = 1/4$$
$$a_2 = a_{-2} = 1/2$$
$$a_3 = a_{-3} = 1/3$$



With these values, the periodic signal $x(t)$ can be re-written as:

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

with, $2\cos(\omega t) = e^{j\omega t} + e^{-j\omega t}$

We obtain,

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

We need to determine the *coefficients* a_k, in order to express a periodic continuous signal $x(t)$ with a fundamental period T and a fundamental frequency $\omega_0 = \frac{2\pi}{T}$ as a *Fourier series*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Find a_k:

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

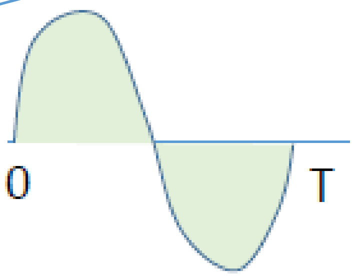
← Multiplying by $e^{-jn\omega_0 t}$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt$$

← Integrating from 0 to T

Using Euler's Formula

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos(k-n)\omega_0 t dt + j\int_0^T \sin(k-n)\omega_0 t dt$$



Area = 0

$$\int_0^T \cos(k-n)\omega_0 t dt = 0, \quad \int_0^T \sin(k-n)\omega_0 t dt = 0$$

← For $k \neq n$

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos(0)dt + j\int_0^T \sin(0)dt = \int_0^T dt = T = T\,\delta_{kn}$$

← For $k = n$ ←

Kronecker Delta:

$$\delta_{kn} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

Fourier Series Representation : Continued

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k T\,\delta_{kn} = a_n T \Rightarrow a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

$$\Rightarrow \boxed{a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt}$$

a_k are called *Fourier series coefficients*, or *spectral coefficients*

Synthesis equation: $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

Analysis equation: $$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_0^T x(t)e^{-jk(2\pi/T)t}dt$$

a_0: is the DC component

Fourier Series Representation : Example 1

A CT signal with fundamental frequency ω_0: $x(t) = sin(\omega_0 t)$, determine its Fourier series

Using Euler's formula: $\begin{array}{l} e^{j\theta} = \cos\theta + j\sin\theta \\ e^{-j\theta} = \cos\theta - j\sin\theta \end{array} \Rightarrow \sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$

$$x(t) = \sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} \qquad (1)$$

Comparing with Fourier synthesis equation(matching terms of (1) and (2)):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \cdots + a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \cdots \qquad (2)$$

We get,

$$a_0 = 0;$$
$$a_1 = (1/2j) = \frac{1}{2j} \times \frac{j}{j} = -\frac{1}{2}j$$
$$a_{-1} = (-1/2j) = \frac{-1}{2j} \times \frac{j}{j} = \frac{1}{2}j$$

and $a_k = 0, \ \text{for } |k| > 1$

Fourier Series Representation : Example 2

Determine Fourier series of: $x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \pi/4)$

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \frac{2}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

$$= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\pi/4}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\pi/4}\right)e^{-j2\omega_0 t}$$

Comparing with Fourier series expansion,

$$x(t) = a_0 + a_1 e^{j\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_{-2}e^{-j2\omega_0 t}$$

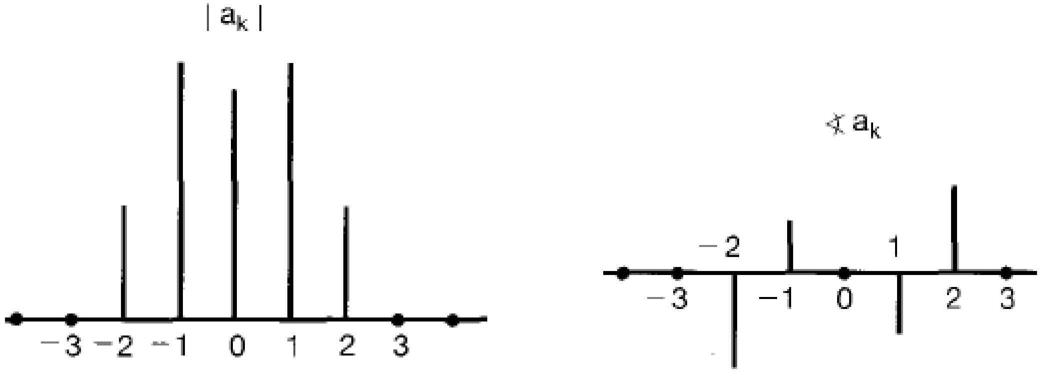
$$a_0 = 1$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j; \quad a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j$$

$$a_2 = \frac{1}{2}e^{j\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) = \frac{1}{2\sqrt{2}}(1+j)$$

$$a_{-2} = \frac{1}{2}e^{-j\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) = \frac{1}{2\sqrt{2}}(1-j)$$

$$a_k = 0, \quad \text{for } |k| > 2$$

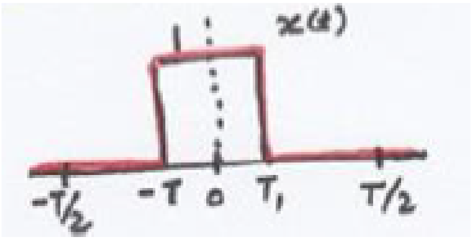


Plots of the magnitude and phase of the Fourier coefficients

Fourier Series Representation : Example 3

Determine Fourier series of periodic *square wave*, defined over one period as:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



Analysis equation:

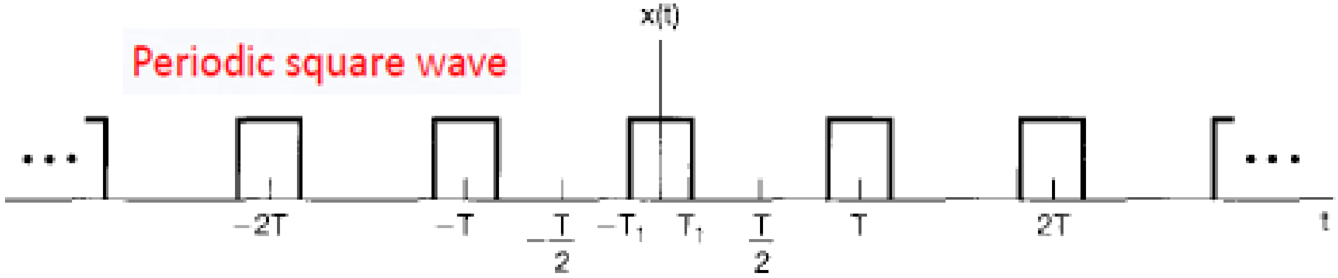
$$a_k = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$

$$a_0 = \frac{1}{T}\int_T x(t) dt = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T} \quad \longleftarrow \; k = 0$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk(2\pi/T)t} dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk(2\pi/T)t} dt$$

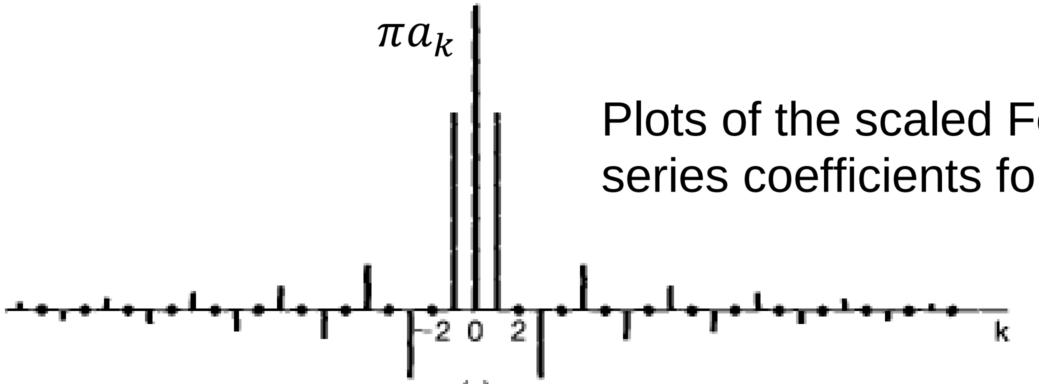
$$= \frac{1}{T} \cdot \frac{-1}{jk(2\pi/T)} e^{-jk(2\pi/T)t} \Big|_{-T_1}^{T_1}$$

$$= \frac{1}{k\pi}\left(\frac{-1}{2j}\right)\left[e^{-jk(2\pi/T)T_1} - e^{jk(2\pi/T)T_1}\right]$$



$$= \left(\frac{1}{k\pi}\right)\left[\frac{e^{jk(2\pi/T)T_1} - e^{-jk(2\pi/T)T_1}}{2j}\right]$$

$$\Rightarrow a_k = \frac{\sin\left(k \frac{2\pi}{T} T_1\right)}{k\pi} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad \text{for } k \neq 0$$

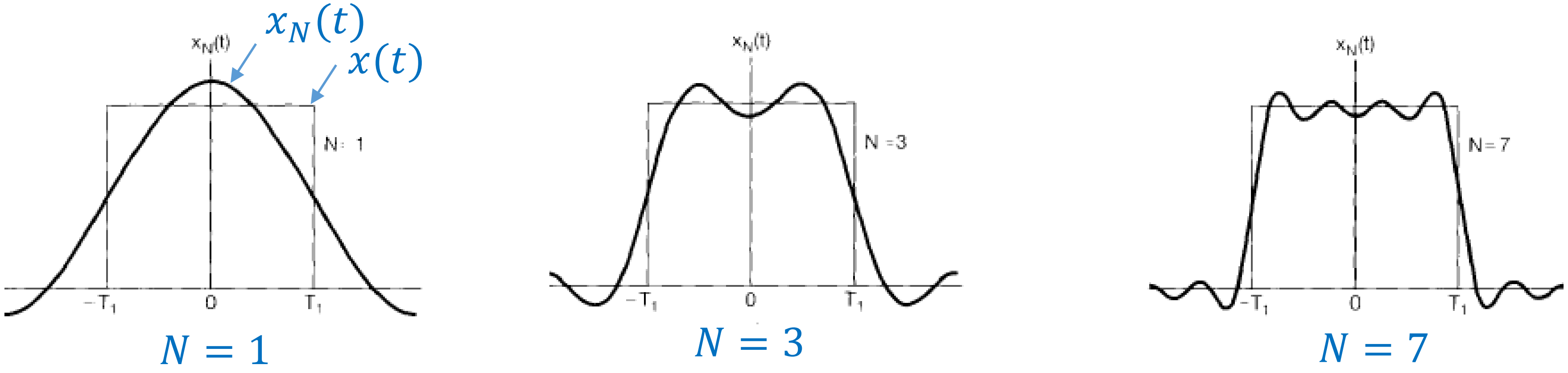


Plots of the scaled Fourier series coefficients for $T = 4T_1$

Fourier series representation of the square wave

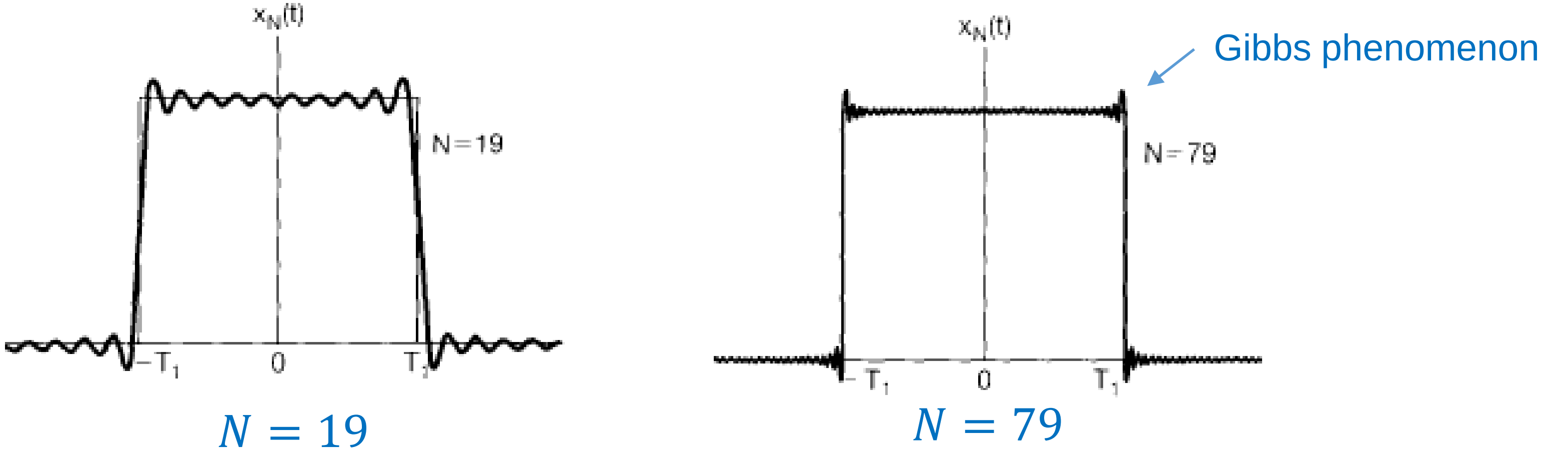
Fourier series can be used to represent (*approximating*) an extremely large class of periodic signals, including the square wave, by a linear combination of a finite number of harmonically related complex exponentials

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}.$$



the approximation error

$$e_N(t) = x(t) - x_N(t) \xrightarrow[N \to \infty]{} 0$$



Convergence of the Fourier series representation of a square wave

Fourier Series Representation : Example 4

A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T = 8$. The non-zero Fourier series coefficients for $x(t)$ are $a_1 = a_{-1} = 2$, $a_3 = a_{-3}^* = 4j$.

Express x(t) in the form: $x(t) = \sum_{k=0}^{\infty} A_k cos(\omega_k t + \varphi_k)$ Find $A_k, \omega_k, and\ \varphi_k$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-3}^{3} a_k e^{jk\omega_0 t}$$

$$= a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_3 e^{j3\omega_0 t} + a_{-3} e^{-j3\omega_0 t}$$

$$= 2e^{j\omega_0 t} + 2e^{-j\omega_0 t} + 4je^{j3\omega_0 t} - 4je^{-j3\omega_0 t}$$

$$= 4\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] + 8(-1)\left[\frac{e^{j3\omega_0 t} - e^{-j3\omega_0 t}}{2j}\right]$$

$$= 4\cos(\omega_0 t) - 8\sin(3\omega_0 t)$$

$$= 4\cos(\omega_0 t + 0) - 8\cos(3\omega_0 t + \pi/2)$$

$$A_1 = 4;\ \omega_1 = \omega_0 = \frac{2\pi}{8} = \frac{\pi}{4};\ \varphi_1 = 0$$

$$A_3 = -8;\ \omega_3 = 3\omega_0 = 3\frac{2\pi}{8} = \frac{3\pi}{4};\ \varphi_1 = \frac{\pi}{2}$$

All other $A_k, \omega_k, \varphi_k = 0$

Fourier Series Representation : Example 5

For the continuous-time periodic signal, $x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$

Determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) = 2 + (1/2)\left[e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}\right] + (4/2j)\left[e^{j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t}\right]$$

$$= 2 + (1/2)e^{j\frac{2\pi}{3}t} + (1/2)e^{-j\frac{2\pi}{3}t} + (-2j)e^{j\frac{5\pi}{3}t} + (2j)e^{-j\frac{5\pi}{3}t}$$

$$x(t) = a_0 + a_1 e^{j\omega_0 t} + a_{-1}e^{-j\omega_0 t}$$
$$+ a_2 e^{j2\omega_0 t} + a_{-2}e^{-j2\omega_0 t}$$
$$+ a_3 e^{j3\omega_0 t} + a_{-3}e^{-j3\omega_0 t}$$
$$+ a_4 e^{j4\omega_0 t} + a_{-4}e^{-j4\omega_0 t}$$
$$+ a_5 e^{j5\omega_0 t} + a_{-5}e^{-j5\omega_0 t} + \cdots$$

$$\omega_0 = \frac{\pi}{3}, a_0 = 2$$
$$a_2 = 1/2, \quad a_{-2} = 1/2$$
$$a_5 = -2j, \quad a_{-5} = 2j$$
$$\text{for all other } k, a_k = 0$$

Properties of Continuous-Time Fourier Series

Fourier series representations possess a number of important properties that are useful for reducing the complexity of the evaluation of the Fourier series of many signals.

For a periodic signal $x(t)$ with period T and fundamental frequency $\omega_0 = 2\pi/T$

Periodic signal $x(t) \overset{FS}{\leftrightarrow} a_k$ Fourier series coefficients

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$	**Synthesis Equation**
$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$	**Analysis Equation**

Properties of Continuous-Time Fourier Series

1 Linearity: two periodic signals $x(t)$ and $y(t)$ With same period T_0

$x(t) \overset{FS}{\leftrightarrow} a_k$

$y(t) \overset{FS}{\leftrightarrow} b_k$

$\Rightarrow$ $Ax(t) + By(t) \overset{FS}{\leftrightarrow} A\,a_k + B\,b_k$

2 Time-Shifting: $x(t) \overset{FS}{\leftrightarrow} a_k$ $\Rightarrow$ $x(t - t_0) \overset{FS}{\leftrightarrow} e^{-jk\omega_0 t_0} a_k \quad (= e^{-jk\left(\frac{2\pi}{T}\right)t_0} a_k)$ $\quad |b_k| = |a_k|.$

$t - t_0 = \tau$

$$b_k = \int_T x(t - t_0)e^{-jk\omega_0 t}\,dt = \int_T x(\tau)e^{-jk\omega_0(\tau + t_0)}d\tau = e^{-jk\omega_0 t_0}\int_T x(\tau)e^{-jk\omega_0 \tau}d\tau = e^{-jk\omega_0 t_0}\,a_k$$

Same magnitudes

3 Frequency-Shifting: $x(t) \overset{FS}{\leftrightarrow} a_k$ $\Rightarrow$ $e^{jM\omega_0 t_0}\,x(t) \overset{FS}{\leftrightarrow} a_{k-M}$

4 Time-Reversal: $x(t) \overset{FS}{\leftrightarrow} a_k$ $\Rightarrow$ $x(-t) \overset{FS}{\leftrightarrow} a_{-k}$ $\Rightarrow$

$x(-t) = x(t)$ $\Rightarrow$ $a_{-k} = a_k$ — *a_k are real even*

$x(-t) = -x(t)$ $\Rightarrow$ $a_{-k} = -a_k$ — *a_k are imaginary and odd*

5 Time-Scaling: $x(t)$ *Period T and frequency ω_0*

$\Rightarrow$ $x(\alpha t)$ *Period T/α and frequency $\alpha\omega_0$*

$x(t) \overset{FS}{\leftrightarrow} a_k$ $\Rightarrow$ $x(\alpha\, t) \overset{FS}{\leftrightarrow} a_k$

6 Multiplication:

two periodic signals $x(t)$ and $y(t)$

$x(t) \overset{FS}{\leftrightarrow} a_k$

$y(t) \overset{FS}{\leftrightarrow} b_k$

$\Rightarrow$ $x(t)\,y(t) \overset{FS}{\leftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}$ $\Leftarrow$ *proof*

$$x(t)y(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \sum_{k=-\infty}^{+\infty} b_k e^{jl\omega_0 t}$$

$$= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} a_k b_l e^{j(k+l)\omega_0 t}$$

$$= \sum_{m=-\infty}^{+\infty}\left[\sum_{l=-\infty}^{+\infty} a_{m-l} b_l\right] e^{jm\omega_0 t}$$

6 Parseval's Relation for Periodic Signals:

The average power of $x(t)$

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \longleftarrow \text{power of } x(t)$$

7 Differentiation:

$$\frac{dx(t)}{dt} \overset{FS}{\leftrightarrow} jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$$

proof

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right] = \sum_{k=-\infty}^{\infty} jk\omega_0\, a_k e^{jk\omega_0 t}$$

8 Integration:

$$x(t) \overset{FS}{\leftrightarrow} a_k \Rightarrow \int_{-\infty}^{t} x(t)\,dt \overset{FS}{\leftrightarrow} \left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{T}{jk2\pi}\right)a_k \qquad k \neq 0$$

9 Conjugate and conjugate Symmetry for real signals:

$$x(t) \overset{FS}{\leftrightarrow} a_k \Rightarrow x^*(t) \overset{FS}{\leftrightarrow} a_{-k}^*$$

$x(t)$ is real

$$x(t) = x^*(t) \Rightarrow \begin{array}{l} a_{-k}^* = a_k \\ |a_k| = |a_{-k}| \end{array}$$

proof

$$a_k = \frac{1}{T_0}\int x(t)e^{-jk\omega_0 t} \Rightarrow a_k^* = \frac{1}{T_0}\int x^*(t)e^{+jk\omega_0 t} \overset{k=-k}{\Rightarrow} a_{-k}^* = \frac{1}{T_0}\int x^*(t)e^{-jk\omega_0 t} \Rightarrow x^*(t) \overset{FS}{\leftrightarrow} a_{-k}^*$$

Conjugate of $x(t)$

10 Periodic Convolution:

$$x(t) \overset{FS}{\leftrightarrow} a_k \text{ and } \quad y(t) \overset{FS}{\leftrightarrow} b_k$$

$$\Rightarrow \quad x(t) * y(t) \overset{FS}{\leftrightarrow} Ta_k b_k$$

proof

$$\frac{1}{T}\int [x(t)*y(t)]e^{-jk\omega_0 t}dt = \frac{1}{T}\int \left[\int_0^T x(\tau)y(t-\tau)\,d\tau\right]e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_0^T\int_0^T x(\tau)y(t-\tau)\frac{e^{jk\omega_0\tau}}{e^{jk\omega_0\tau}}d\tau\, e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_0^T\int_0^T x(\tau)e^{-jk\omega_0\tau}y(t-\tau)d\tau\, e^{jk\omega_0\tau}\, e^{-jk\omega_0 t}dt$$

$t-\tau = m \to dt = dm$

$t = 0 \to m = -\tau$

$t = T \to m = T-\tau$

$$= \frac{1}{T}\left(\int_0^T x(\tau)e^{-jk\omega_0\tau}d\tau\right)\left(\int_0^T y(t-\tau)e^{-jk\omega_0(t-\tau)}dt\right) = \left(\frac{1}{T}\int_0^T x(\tau)e^{-jk\omega_0\tau}d\tau\right)\left(\frac{1}{T}T\int_{-\tau}^{T-\tau} y(m)e^{-jk\omega_0(m)}dm\right) = Ta_k b_k$$

Problem 1

Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk(2\pi/50)t}$$

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk(2\pi/50)t}$$

$$x_3(t) = \sum_{k=-100}^{100} j\sin(k\pi/2) e^{jk(2\pi/50)t}$$

Use Fourier series properties to help answer the following questions:

(a) Which of the three signals is/are even?
(b) Which of the three signals is/are real valued?

Fourier series representation: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

For $x_1(t)$ $\quad \omega_0 = \frac{2\pi}{50}$

For $x_1(t)$ to be real: $a_{-k}^* = a_k$

$$a_k = \left(\frac{1}{2}\right)^k, for \;\; k = 0, 1, 2, \cdots, 100$$

$$a_k = 0, for\ k > 100\ and\ k < 0$$

However, here $a_{10} = \left(\frac{1}{2}\right)^{10}$

$$a_{-10} = 0, \qquad a_{10} \neq a_{-10}^*$$

$x_1(t)$ is not real.

For $x_1(t)$ to be even: $x_1(t) = x_1(-t)$

$$x_1(-t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{-jk\left(\frac{2\pi}{50}\right)t} = \sum_{k=-100}^{0} \left(\frac{1}{2}\right)^{-k} e^{jk\left(\frac{2\pi}{50}\right)t} \neq \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\left(\frac{2\pi}{50}\right)t} = x_1(t)$$

$x_1(t)$ is not even.

For $x_2(t)$

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk(2\pi/50)t}$$

$$\omega_0 = \frac{2\pi}{50}$$

$a_k = cos(k\pi), for\ -100 \le k \le 100$

$a_k = 0, \quad otherwise$

For $x_1(t)$ to be real: $a^*_{-k} = a_k$

$$a^*_{-k} = \left(cos(-k\pi)\right)^* = cos(k\pi) = a_k$$

$$Re\{a_k\} = cos(k\pi), \qquad Re\{a_{-k}\} = cos(-k\pi) = cos(k\pi)$$

$$Re\{a_k\} = Re\{a_{-k}\}$$

$$Im\{a_k\} = 0 = Im\{a_{-k}\}$$

$$|a_k| = |a_{-k}|, \qquad \measuredangle a_k = 0 = \measuredangle a_{-k}$$

$x_2(t)$ is real.

For $x_1(t)$ to be even: $x_1(t) = x_1(-t)$, and $a_k = a_{-k}$

$$a_k = cos(k\pi)$$

$$a_{-k} = cos(-k\pi) = cos(k\pi)$$

$$a_k = a_{-k}$$

$x_2(t)$ is even.

For $x_3(t)$

$$x_3(t) = \sum_{k=-100}^{100} j \sin(k\pi/2) e^{jk(2\pi/50)t}$$

$$\omega_0 = \frac{2\pi}{50}$$

$a_k = j\ sin(k\pi/2), for\ -100 \le k \le 100$

$a_k = 0, \quad otherwise$

For $x_1(t)$ to be real: $a^*_{-k} = a_k$

$a_k = j\ sin(k\pi/2)$

$$a^*_{-k} = -j\ sin\left(-\frac{k\pi}{2}\right) = j\ sin\left(\frac{k\pi}{2}\right) = a_k$$

$\Rightarrow$ $Re\{a_k\} = 0 = Re\{a_{-k}\}$

$\Rightarrow$ $Im\{a_k\} = -Im\{a_{-k}\}$

$|a_k| = |j\ sin(k\pi/2)| = |sin(k\pi/2)|$

$|a_{-k}| = |j\ sin(-k\pi/2)| = |sin(k\pi/2)|$

$\Rightarrow$ $|a_k| = |a_{-k}|$

$$\measuredangle a_k = tan^{-1}\left(\frac{sin(k\pi/2)}{0}\right) = tan^{-1}(\infty) = \pi/2$$

$$\measuredangle a_{-k} = tan^{-1}\left(\frac{sin(-k\pi/2)}{0}\right) = tan^{-1}(-\infty) = -\pi/2$$

$\Rightarrow$ $\measuredangle a_k = -\measuredangle a_{-k}$

$x_3(t)$ is real.

For $x_1(t)$ to be even: $a_k = a_{-k}$

$$a_{-k} = j\ sin\left(-\frac{k\pi}{2}\right) = -jsin\left(\frac{k\pi}{2}\right) = -a_k$$

$\Rightarrow$ $a_{-k} \neq a_k$

$x_3(t)$ is not even.

Problem 2

Suppose we are given the following information about a signal $x(t)$:

(1) $x(t)$ is real and odd.

(2) $x(t)$ is periodic with period $\mathrm{T} = 2$, and has Fourier coefficients a_k:

(3) $a_k = 0$ for $|k| > 1$.

(4) $\frac{1}{2}\int_0^2 |x(t)|^2\, dt = 1$.

Specify a signal that satisfies these conditions.

From (2): Fourier series representation: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

From (3): $x(t) = a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$

From (1): $a_0 = 0$, because $x(t)$ is odd.

$x(t)$ is odd ➡ $a_1 = -a_{-1}$

$$x(t) = a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$$
$$= a_1\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

➡ $x^*(t) = a_1^*\left(e^{-j\omega_0 t} - e^{j\omega_0 t}\right)$

➡ $|x(t)|^2 = |x(t)\, x^*(t)| = |a_1\, a_1^*|\left|1 - e^{j\omega_0 t} - e^{-j\omega_0 t} + 1\right|$

$$= |a_1\, a_1^*||2 - 2cos(2\omega_0 t)| = 2|a_1\, a_1^*||1 - cos(2\omega_0 t)|$$

From (4): $\frac{1}{2}\int_0^2 |x(t)|^2\, dt = 1.$ ⇒ $\frac{1}{2}\int_0^2 2|a_1\, a_1^*||1 - cos(2\omega_0 t)|dt = 1.$

⇒ $|a_1\, a_1^*| \int_0^2 |1 - cos(2\omega_0 t)|dt = 1.$

$$|a_1\, a_1^*| \left[t - \frac{\sin(2\omega_0 t)}{2\omega_0}\right]_0^2 = 1.$$ ⇒ $|a_1\, a_1^*|\, [2 - 0 - 0 + 0] = 1$ ⇒ $|a_1\, a_1^*| = \frac{1}{2}$

As a_1 is complex: $\{Re(a_1)\}^2 + \{Im(a_1)\}^2 = \frac{1}{2}$

As a_1 is purely imaginary: $\{0\}^2 + \{Im(a_1)\}^2 = \frac{1}{2}$ ⇒ $Im(a_1) = \pm\frac{1}{\sqrt{2}}$

$$a_1 = \pm j\frac{1}{\sqrt{2}}$$

$$a_{-1} = -a_1 = \mp j\frac{1}{\sqrt{2}}$$

Therefore, the signals are

$$x_1(t) = \frac{1}{\sqrt{2}}je^{j\pi t} - \frac{1}{\sqrt{2}}je^{-j\pi t} = -\sqrt{2}\sin(\pi t)$$

$$x_2(t) = -\frac{1}{\sqrt{2}}je^{j\pi t} + \frac{1}{\sqrt{2}}je^{-j\pi t} = \sqrt{2}\sin(\pi t)$$