PHYSICS 502
3rd HOMEWORK
Dr. V. Lempesis

Hand in: Sunday 14th April 2013

1. When a single mode laser field with a frequency ω_L, which is very close to the transition frequency ω_0 between two certain atomic levels then the atom can be considered to a very good approximation as a *two-level* atom (see figure below). In this case the quantum mechanical density matrix is the 2×2 matrix:

$$\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

where the diagonal elements ρ_{11}, ρ_{22} represents the populations of the two states 1 and 2, thus:

$$\rho_{11}(t)+\rho_{22}(t)=1\,. \qquad (1)$$

The off-diagonal elements represent the so called *coherences* and they satisfy the condition $\rho_{12}=\rho_{21}^{*}$. The density matrix elements, after some elaboration, satisfy the following equations, known as *optical Bloch equations*:

$$\frac{d\rho_{22}(t)}{dt}=G\rho_{12}(t)+G^{*}\rho_{21}(t) \qquad (2)$$

$$\frac{d\rho_{11}(t)}{dt}=-G\rho_{12}(t)-G^{*}\rho_{21}(t) \qquad (3)$$

$$\frac{d\rho_{12}(t)}{dt}=G^{*}\rho_{11}(t)+G\rho_{22}(t)+i\Delta\rho_{12}(t) \qquad (4)$$

$$\frac{d\rho_{21}(t)}{dt}=G\rho_{11}(t)+G^{*}\rho_{22}(t)-i\Delta\rho_{12}(t) \qquad (5)$$

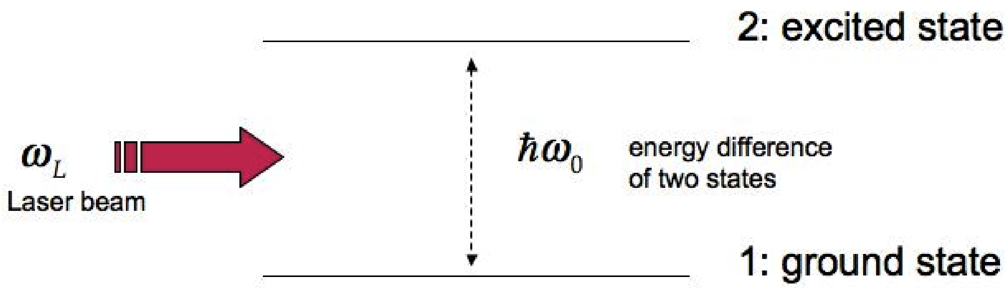
Where $\Delta=\omega_L-\omega_0$ is the so called *detuning* and G is the so called *Rabi frequency*, which determines the strength of the interaction between the laser and the atom.

(a) Apply Laplace transforms to equations (1) to (5).

(b) Solve the system which you get from question (a) and get the expressions for the Laplace transformed density matrix elements, $\rho_{11}(s)$, $\rho_{22}(s)$, $\rho_{12}(s)$ and $\rho_{21}(s)$.

(c) Apply inverse Laplace transforms and get the time dependent form of the density matrix elements.

Hint: In your workings consider that the Rabi frequency G is real, that the atom initially is at the ground state ($\rho_{11}(0)=1$) and that the initial coherences are zero, i.e. $\rho_{12}(0)=\rho_{21}(0)=0$.



Solution:

The Laplace transformed equations of (2)-(5) are

$$s\rho_{22}(s)-\rho_{22}(0)=G\rho_{12}(s)+G\rho_{21}(s) \qquad (6)$$

$$s\rho_{11}(s)-\rho_{11}(0)=-G\rho_{12}(s)-G\rho_{21}(s) \qquad (7)$$

$$s\rho_{12}(s)-\rho_{12}(0)=G\rho_{11}(s)+G\rho_{22}(s)+i\Delta\rho_{12}(s) \qquad (8)$$

$$s\rho_{21}(s)-\rho_{21}(0)=G\rho_{11}(s)+G\rho_{22}(s)-i\Delta\rho_{12}(s) \qquad (9)$$

Since

$$\rho_{11}(t)+\rho_{22}(t)=1 \Rightarrow \rho_{11}(s)+\rho_{22}(s)=1/s \Rightarrow$$
$$\rho_{22}(s)=1/s-\rho_{11}(s)$$

Substituting this in (7), (8) and (9) we get the following system of three equations with three unknown quantities:

$$s\rho_{11}(s)+G\rho_{12}(s)+G\rho_{21}(s)=\rho_{11}(0) \qquad (10)$$

$$s\rho_{12}(s) - G\rho_{11}(s) - G\big(1/s - \rho_{11}(s)\big) - i\Delta\rho_{12}(s) = \rho_{12}(0) \quad (11)$$

$$s\rho_{21}(s) - G\rho_{11}(s) - G\big(1/s - \rho_{11}(s)\big) + i\Delta\rho_{12}(s) = \rho_{21}(0) \quad (12)$$

Which are transformed as follows (remember that $\rho_{12}(0) = \rho_{21}(0) = 0$ and $\rho_{11}(0) = 1$).

$$s\rho_{11}(s) + G\rho_{12}(s) + G\rho_{21}(s) = 1 \qquad (13)$$

$$(s - i\Delta)\rho_{12}(s) = G/s \quad (14)$$

$$(s + i\Delta)\rho_{21}(s) = G/s \quad (15)$$

From (14) and (15) we get the following expressions:

$$\rho_{12}(s) = G/\big[s(s - i\Delta)\big] \quad (16)$$

$$\rho_{21}(s) = G/\big[s(s + i\Delta)\big] \quad (17)$$

By working partial fractions we have

$$\rho_{12}(s) = G/\big[s(s - i\Delta)\big] = A/s + B/(s - i\Delta)$$

with $A = iG/\Delta,\ B = -iG/\Delta$

by taking the inverse Laplace transform we have

$$\rho_{12}(t) = i\frac{G}{\Delta}\left\{1 - e^{i\Delta t}\right\} \quad (18)$$

and similarly $\rho_{12}(t) = -i\frac{G}{\Delta}\left\{1 - e^{-i\Delta t}\right\}$ (19).

From equation (13)

$$s\rho_{11}(s) + G\rho_{12}(s) + G\rho_{21}(s) = 1 \Rightarrow s\rho_{11}(s) + G\frac{G}{s(s - i\Delta)} + G\frac{G}{s(s + i\Delta)} = 1 \Rightarrow$$

$$s\rho_{11}(s) + \frac{2G^2}{(s^2 + \Delta^2)} = 1 \Rightarrow \rho_{11}(s) = \frac{1}{s} - \frac{2G^2}{s(s^2 + \Delta^2)} = \frac{s^2 + \Delta^2 - 2G^2}{s(s^2 + \Delta^2)}$$

But this can be written as

$$\rho_{11}(s) = \frac{A}{s} + \frac{Bs + C}{(s^2 + \Delta^2)}$$

$$A = 1 - \frac{2G^2}{\Delta^2}, \quad B = \frac{2G^2}{\Delta^2}, \quad C = 0$$

Thus

$$\rho_{11}(t) = \left(1 - \frac{2G^2}{\Delta^2}\right) + \frac{2G^2}{\Delta^2}\cos(\Delta t) \qquad (20)$$

$$\rho_{22}(t) = 1 - \rho_{11}(t) = \frac{2G^2}{\Delta^2} - \frac{2G^2}{\Delta^2}\cos(\Delta t) \qquad (21)$$