12. Deflections of Beams and Shafts

CHAPTER OBJECTIVES

• Use various methods to determine the deflection and slope at specific pts on beams and shafts:
  1. Integration method
  2. Discontinuity functions
  3. Method of superposition
  4. Moment-area method
12. Deflections of Beams and Shafts

12.1 THE ELASTIC CURVE

- It is useful to sketch the deflected shape of the loaded beam, to “visualize” computed results and partially check the results.

- The deflection diagram of the longitudinal axis that passes through the centroid of each x-sectional area of the beam is called the elastic curve.
12. Deflections of Beams and Shafts

12.1 THE ELASTIC CURVE

- Draw the moment diagram for the beam first before creating the elastic curve.
- Use beam convention as shown and established in chapter 6.1.
12. Deflections of Beams and Shafts

12.1 THE ELASTIC CURVE

- For example, due to roller and pin supports at $B$ and $D$, displacements at $B$ and $D$ is zero.
- For region of -ve moment $AC$, elastic curve concave downwards.
- Within region of +ve moment $CD$, elastic curve concave upwards.
- At pt $C$, there is an inflection pt where curve changes from concave up to concave down (zero moment).
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12.1 THE ELASTIC CURVE

(a) A

(b) Moment diagram

(c) A

Inflection point

Elastic curve

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12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

• $x$ axis extends +ve to the right, along longitudinal axis of beam.

• A differential element of undeformed width $dx$ is located.

• $y$ axis extends +ve upwards from $x$ axis. It measures the displacement of the centroid on $x$-sectional area of element.

• A “localized” $y$ coordinate is specified for the position of a fiber in the element.

• It is measured +ve upward from the neutral axis.
12. Deflections of Beams and Shafts

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- Limit analysis to the case of initially straight beam elastically deformed by loads applied perpendicular to beam’s \( x \) axis and lying in the \( x-\nu \) plane of symmetry for beam’s \( x \)-sectional area.
- Internal moment \( M \) deforms element such that angle between \( x \)-sections is \( d\theta \).
- Arc \( dx \) is a part of the elastic curve that intersects the neutral axis for each \( x \)-section.
- Radius of curvature for this arc defined as the distance \( \rho \), measured from center of curvature \( O' \) to \( dx \).
12. Deflections of Beams and Shafts

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- Strain in arc $ds$, at position $y$ from neutral axis, is

$$\varepsilon = \frac{ds' - ds}{ds}$$

But $ds = dx = \rho d\theta$ and $ds' = (\rho - y)d\theta$

$$\varepsilon = \frac{[(\rho - y)d\theta - \rho d\theta_s]}{\rho d\theta}$$

or

$$\frac{1}{\rho} = -\frac{\varepsilon}{y}$$

(12-1)
12. Deflections of Beams and Shafts

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- If material is homogeneous and shows linear-elastic behavior, Hooke’s law applies. Since flexure formula also applies, we combing the equations to get

\[ \varepsilon = \sigma / E, \sigma = -M y / I \rightarrow \frac{1}{\rho} = \frac{M}{EI} \]  \hspace{1cm} (12 - 2)

\( \rho = \) radius of curvature at a specific pt on elastic curve (1/\( \rho \) is referred to as the curvature).

\( M = \) internal moment in beam at pt where is to be determined.

\( E = \) material’s modulus of elasticity.

\( I = \) beam’s moment of inertia computed about neutral axis.
12. Deflections of Beams and Shafts

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- \( EI \) is the flexural rigidity and is always positive.
- Sign for \( \rho \) depends on the direction of the moment.
- As shown, when \( M \) is +ve, \( \rho \) extends above the beam. When \( M \) is –ve, \( \rho \) extends below the beam.
12. Deflections of Beams and Shafts

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- Using flexure formula, $\sigma = -\frac{My}{I}$, curvature is also

$$\frac{1}{\rho} = -\frac{\sigma}{Ey} \quad (12-3)$$

- Eqns 12-2 and 12-3 valid for either small or large radii of curvature.
12. Deflections of Beams and Shafts

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

• Let’s represent the curvature in terms of \( \nu \) and \( x \).

\[
\frac{1}{\rho} = \frac{d^2\nu}{dx^2} \left[ 1 + \left( \frac{d\nu}{dx} \right)^2 \right]^{3/2}
\]

• Substitute into Eqn 12-2

\[
\frac{d^2\nu}{dx^2} \left[ 1 + \left( \frac{d\nu}{dx} \right)^2 \right]^{3/2} = \frac{M}{EI} \quad (12 - 4)
\]
12. Deflections of Beams and Shafts

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Most engineering codes specify limitations on deflections for tolerance or aesthetic purposes.
- Slope of elastic curve determined from \( \frac{d\nu}{dx} \) is very small and its square will be negligible compared with unity.
- Therefore, by approximation \( \frac{1}{\rho} = \frac{d^2\nu}{dx^2} \), Eqn 12-4 rewritten as
  \[
  \frac{d^2\nu}{dx^2} = \frac{M}{EI} \quad (12 - 5)
  \]
- Differentiate each side w.r.t. \( x \) and substitute \( V = \frac{dM}{dx} \), we get
  \[
  \frac{d}{dx} \left( EI \frac{d^2\nu}{dx^2} \right) = V(x) \quad (12 - 6)
  \]
12. Deflections of Beams and Shafts

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Differentiating again, using \( w = dV/dx \) yields
  \[
  \frac{d^2}{dx^2} \left( EI \frac{d^2 \nu}{dx^2} \right) = -w(x) \quad (12 - 7)
  \]

- Flexural rigidity is constant along beam, thus
  \[
  EI \frac{d^4 \nu}{dx^4} = -w(x) \quad (12 - 8)
  \]
  \[
  EI \frac{d^3 \nu}{dx^3} = V(x) \quad (12 - 9)
  \]
  \[
  EI \frac{d^2 \nu}{dx^2} = M(x) \quad (12 - 10)
  \]
12. Deflections of Beams and Shafts

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Generally, it is easier to determine the internal moment $M$ as a function of $x$, integrate twice, and evaluate only two integration constants.
- For convenience in writing each moment expression, the origin for each $x$ coordinate can be selected arbitrarily.

Sign convention and coordinates
- Use the proper signs for $M$, $V$ and $w$. 

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• Positive deflection $v$ is upward.

• Positive slope angle $\theta$ will be measured counterclockwise when $x$ is positive to the right.

• Positive slope angle $\theta$ will be measured clockwise when $x$ is positive to the left.
12. Deflections of Beams and Shafts

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION
12. Deflections of Beams and Shafts

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Boundary and continuity conditions

- Possible boundary conditions are shown here.

1. \[ \Delta = 0 \]
   Roller

2. \[ \Delta = 0 \]
   Pin

3. \[ \Delta = 0 \]
   Roller

4. \[ \Delta = 0 \]
   Pin

5. \[ \theta = 0 \]
   \[ \Delta = 0 \]
   Fixed end

6. \[ V = 0 \]
   \[ M = 0 \]
   Free end

7. \[ M = 0 \]
   Internal pin or hinge
Boundary and continuity conditions

- If a single $x$ coordinate cannot be used to express the eqn for beam’s slope or elastic curve, then continuity conditions must be used to evaluate some of the integration constants.

\[ 0 \leq x_1 \leq a, \ a \leq x_2 \leq (a+b) \]

\[ \theta_1 (a) = \theta_2 (a) \]
\[ v_1 (a) = v_2 (a) \]

\[ 0 \leq x_1 \leq a, \ 0 \leq x_2 \leq b \]

\[ \theta_1 (a) = -\theta_2 (b) \]
\[ v_1 (a) = v_2 (b) \]
12. Deflections of Beams and Shafts

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Elastic curve

- Draw an exaggerated view of the beam’s elastic curve.
- Recall that zero slope and zero displacement occur at all fixed supports, and zero displacement occurs at all pin and roller supports.
- Establish the $x$ and $y$ coordinate axes.
- The $x$ axis must be parallel to the undeflected beam and can have an origin at any point along the beam, with $+$ve direction either to the right or to the left.
Procedure for analysis

Elastic curve

• If several discontinuous loads are present, establish $x$ coordinates that are valid for each region of the beam between the discontinuities.

• Choose these coordinates so that they will simplify subsequent algebraic work.
12. Deflections of Beams and Shafts

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Load or moment function

- For each region in which there is an \( x \) coordinate, express that loading \( w \) or the internal moment \( M \) as a function of \( x \).
- In particular, always assume that \( M \) acts in the +ve direction when applying the eqn of moment equilibrium to determine \( M = f(x) \).
12. Deflections of Beams and Shafts

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Slope and elastic curve

• Provided $EI$ is constant, apply either the load eqn
  $EI \frac{d^4 \nu}{dx^4} = -w(x)$, which requires four integrations to get
  $\nu = \nu(x)$, or the moment eqns
  $EI \frac{d^2 \nu}{dx^2} = M(x)$, which requires only two integrations. For each integration, we include a
  constant of integration.

• Constants are evaluated using boundary conditions for the supports and the continuity
  conditions that apply to slope and displacement at
  pts where two functions meet.
Procedure for analysis

Slope and elastic curve

• Once constants are evaluated and substituted back into slope and deflection eqns, slope and displacement at specific pts on elastic curve can be determined.

• The numerical values obtained is checked graphically by comparing them with sketch of the elastic curve.

• Realize that +ve values for slope are counterclockwise if the $x$ axis extends +ve to the right, and clockwise if the $x$ axis extends +ve to the left. For both cases, +ve displacement is upwards.
EXAMPLE 12.1

Cantilevered beam shown is subjected to a vertical load $P$ at its end. Determine the eqn of the elastic curve. $EI$ is constant.
Elastic curve: Load tends to deflect the beam. By inspection, the internal moment can be represented throughout the beam using a single $x$ coordinate.

Moment function: From free-body diagram, with $M$ acting in the $+ve$ direction, we have

$$M = -Px$$
Slope and elastic curve:
Applying Eqn 12-10 and integrating twice yields

\[ EI \frac{d^2 \nu}{dx^2} = -Px \]  \hspace{1cm} (1)

\[ EI \frac{d \nu}{dx} = -\frac{Px^2}{2} + C_1 \]  \hspace{1cm} (2)

\[ EI \nu = -\frac{Px^3}{6} + C_1 x + C_2 \]  \hspace{1cm} (3)
Slope and elastic curve:

Using boundary conditions $\frac{dv}{dx} = 0$ at $x = L$, and $v = 0$ at $x = L$, Eqn (2) and (3) becomes

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1L + C_2$$
EXAMPLE 12.1 (SOLN)

Slope and elastic curve:
Thus, \( C_1 = \frac{PL^2}{2} \) and \( C_2 = \frac{PL^3}{3} \). Substituting these results into Eqns (2) and (3) with \( \theta = \frac{d\nu}{dx} \), we get

\[
\theta = -\frac{P}{2EI} \left( L^2 - x^2 \right)
\]

\[
\nu = \frac{P}{6EI} \left( -x^3 + 3L^2x - 2L^3 \right)
\]

Maximum slope and displacement occur at \( A \) \((x = 0)\),

\[
\theta_A = \frac{PL^2}{2EI} \quad \quad \nu_A = -\frac{PL^3}{3EI}
\]
Slope and elastic curve:
Positive result for $\theta_A$ indicates counterclockwise rotation and negative result for $\nu_A$ indicates that $\nu_A$ is downward.

Consider beam to have a length of 5 m, support load $P = 30$ kN and made of A-36 steel having $E_{st} = 200$ GPa.
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EXAMPLE 12.1 (SOLN)

Slope and elastic curve:
Using methods in chapter 11.3, assuming allowable normal stress is equal to yield stress $\sigma_{allow} = 250$ MPa, then a W310×39 would be adequate ($I = 84.8(10^6)$ mm$^4$).

From Eqns (4) and (5),

$$\theta_A = \frac{PL^2}{2EI} \quad \nu_A = -\frac{PL^3}{3EI}$$
Slope and elastic curve:

From Eqns (4) and (5),

\[
\theta_A = \frac{30 \text{ kN} \left(10^3 \text{ N/kN}\right) \times \left[5 \text{ m} \left(10^3 \text{ mm/m}\right)^2\right]^2}{2 \left[200 \left(10^3 \right) \text{ N/mm}^2 \right] \left[84.8 \left(10^6 \right) \text{ mm}^4\right]} = 0.0221 \text{ rad}
\]

\[
\nu_A = -\frac{30 \text{ kN} \left(10^3 \text{ N/kN}\right) \times \left[5 \text{ m} \left(10^3 \text{ mm/m}\right)^2\right]^3}{3 \left[200 \left(10^3 \right) \text{ N/mm}^2 \right] \left[84.8 \left(10^6 \right) \text{ mm}^4\right]} = -73.7 \text{ mm}
\]
Slope and elastic curve:
Since \( \theta_A^2 = (d \nu/dx)^2 = 0.000488 \ll 1 \), this justifies the use of Eqn 12-10 than the more exact 12-4.
SOLUTION 2

Using Eqn 12-8 to solve the problem. Here \( w(x) = 0 \) for \( 0 \leq x \leq L \), so that upon integrating once, we get the form of Eqn 12-19

\[
EI \frac{d^4 \nu}{dx^4} = 0
\]

\[
EI \frac{d^3 \nu}{dx^3} = C'_1 = V
\]
Solution II

Shear constant $C'_1$ can be evaluated at $x = 0$, since $V_A = -P$. Thus, $C'_1 = -P$. Integrating again yields the form of Eqn 12-10,

$$EI \frac{d^3 \nu}{dx^3} = -P$$

$$EI \frac{d^2 \nu}{dx^2} = -Px + C'_2 = M$$

Here, $M = 0$ at $x = 0$, so $C'_2 = 0$, and as a result, we obtain Eqn 1 and solution proceeds as before.
The simply supported beam shown in Fig. 12–11a supports the triangular distributed loading. Determine its maximum deflection. $EI$ is constant.
12. Deflections of Beams and Shafts

Solution I

Elastic Curve. Due to symmetry, only one $x$ coordinate is needed for the solution, in this case $0 \leq x \leq L/2$. The beam deflects as shown in Fig. 12–11a. Notice that maximum deflection occurs at the center since the slope is zero at this point.

Moment Function. The distributed load acts downward, and therefore, it is positive according to our sign convention. A free-body diagram of the segment on the left is shown in Fig. 12–11b. The equation for the distributed loading is

$$w = \frac{2w_0}{L}x$$

Hence,

$$\sum M_{NA} = 0; \quad M + \frac{w_0 x^2}{L} \left( \frac{x}{3} \right) - \frac{w_0 L}{4} (x) = 0$$

$$M = -\frac{w_0 x^3}{3L} + \frac{w_0 L}{4} x$$
12. Deflections of Beams and Shafts

Slope and Elastic Curve.

Using Eq. 12–10 and integrating twice, we have

\[
EI \frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x
\]

\[
EI \frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 + C_1
\]

\[
EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 + C_1x + C_2
\]

The constants of integration are obtained by applying the boundary condition \( v = 0 \) at \( x = 0 \) and the symmetry condition that \( dv/dx = 0 \) at \( x = L/2 \). This leads to

\[
C_1 = -\frac{5w_0L^3}{192} \quad C_2 = 0
\]

Hence,

\[
EI \frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 - \frac{5w_0L^3}{192}
\]

\[
EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 - \frac{5w_0L^3}{192}x
\]

Determining the maximum deflection at \( x = L/2 \), we have

\[
v_{\text{max}} = -\frac{w_0L^4}{120EI}
\]

\( \text{Ans.} \)
12. Deflections of Beams and Shafts

**Example 12.3**

The simply supported beam shown in Fig. 12–12a is subjected to the concentrated force $P$. Determine the maximum deflection of the beam. $EI$ is constant.

Note: This Example is solved with Length and Force as Variable.
12. Deflections of Beams and Shafts

**Elastic Curve.** The beam deflects as shown in Fig. 12–12b. Two coordinates must be used, since the moment becomes discontinuous at $P$. Here we will take $x_1$ and $x_2$, having the *same origin* at $A$, so that $0 \leq x_1 < 2a$ and $2a < x_2 \leq 3a$.

**Moment Function.** From the free-body diagrams shown in Fig. 12–12c,

\[
M_1 = \frac{P}{3} x_1
\]

\[
M_2 = \frac{P}{3} x_2 - P(x_2 - 2a) = \frac{2P}{3} (3a - x_2)
\]

**Slope and Elastic Curve.** Applying Eq. 12–10 for $M_1$ and integrating twice yields

\[
EI \frac{d^2v_1}{dx_1^2} = \frac{P}{3} x_1
\]

\[
EI \frac{dv_1}{dx_1} = \frac{P}{6} x_1^2 + C_1
\]

\[
EI v_1 = \frac{P}{18} x_1^3 + C_1 x_1 + C_2
\]

Likewise for $M_2$,

\[
EI \frac{d^2v_2}{dx_2^2} = \frac{2P}{3} (3a - x_2)
\]

\[
EI \frac{dv_2}{dx_2} = \frac{2P}{3} \left( 3ax_2 - \frac{x_2^2}{2} \right) + C_3
\]

\[
EI v_2 = \frac{2P}{3} \left( \frac{3}{2} ax_2^2 - \frac{x_2^3}{6} \right) + C_3 x_2 + C_4
\]
The four constants are evaluated using two boundary conditions, namely, $x_1 = 0$, $v_1 = 0$ and $x_2 = 3a$, $v_2 = 0$. Also, two continuity conditions must be applied at $B$, that is, $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$ and $v_1 = v_2$ at $x_1 = x_2 = 2a$. Substitution as specified results in the following four equations:

$v_1 = 0$ at $x_1 = 0$; \hspace{1cm} 0 = 0 + 0 + C_2

$v_2 = 0$ at $x_2 = 3a$; \hspace{1cm} 0 = \frac{2P}{3} \left( \frac{3}{2} a(3a)^2 - \frac{(3a)^3}{6} \right) + C_3(3a) + C_4

\frac{dv_1(2a)}{dx_1} = \frac{dv_2(2a)}{dx_2}; \hspace{1cm} \frac{P}{6}(2a)^2 + C_1 = \frac{2P}{3} \left( 3a(2a) - \frac{(2a)^2}{2} \right) + C_3

v_1(2a) = v_2(2a); \hspace{1cm} \frac{P}{18}(2a)^3 + C_1(2a) + C_2 = \frac{2P}{3} \left( \frac{3}{2} a(2a)^2 - \frac{(2a)^3}{6} \right) + C_3(2a) + C_4

Solving these equations, we get

$C_1 = -\frac{4}{9} Pa^2$ \hspace{1cm} $C_2 = 0$

$C_3 = -\frac{22}{9} Pa^2$ \hspace{1cm} $C_4 = \frac{4}{3} Pa^3$
Thus, Eqs. 1–4 become

\[
\frac{dv_1}{dx_1} = \frac{P}{6EI} x_1^2 - \frac{4Pa^2}{9EI} \tag{5}
\]

\[
v_1 = \frac{P}{18EI} x_1^3 - \frac{4Pa^2}{9EI} x_1 \tag{6}
\]

\[
\frac{dv_2}{dx_2} = \frac{2Pa}{EI} x_2 - \frac{P}{3EI} x_2^2 - \frac{22Pa^2}{9EI} \tag{7}
\]

\[
v_2 = \frac{Pa}{EI} x_2^2 - \frac{P}{9EI} x_2^3 - \frac{22Pa^2}{9EI} x_2 + \frac{4Pa^3}{3EI} \tag{8}
\]

By inspection of the elastic curve, Fig. 12–12b, the maximum deflection occurs at \(D\), somewhere within region \(AB\). Here the slope must be zero. From Eq. 5,

\[
\frac{1}{6} x_1^2 - \frac{4}{9} a^2 = 0
\]

\[
x_1 = 1.633a
\]

Substituting into Eq. 6,

\[
\nu_{\text{max}} = -0.484 \frac{Pa^3}{EI}
\]

\(\text{Ans.}\)

The negative sign indicates that the deflection is downward.
EXAMPLE 12.4

The Beam in Fig. is subjected to a load at its end. Determine the displacement at $C$. $EI$ is constant.
Elastic Curve. The beam deflects into the shape shown in Fig. 12–13a. Due to the loading, two $x$ coordinates will be considered, namely, $0 \leq x_1 < 2$ m and $0 \leq x_2 < 1$ m, where $x_2$ is directed to the left from C, since the internal moment is easy to formulate.
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EXAMPLE 12.4 (SOLN)

Moment functions

Using free-body diagrams, we have

\[ M_1 = -2x_1 \quad M_2 = -4x_2 \]

Slope and Elastic curve: Applying Eqn 10-12,

For \( 0 \leq x_1 \leq 2 \):

\[ EI \frac{d^2 v_1}{dx_1^2} = -2x_1 \]

\[ EI \frac{dv_1}{dx_1} = -x_1^2 + C_1 \quad \text{(1)} \]

\[ EI v_1 = -\frac{1}{3}x_1^3 + C_1 x_1 + C_2 \quad \text{(2)} \]
Slope and Elastic curve:
Applying Eqn 10-12,

For $0 \leq x_2 \leq 1$ m: \[ EI \frac{d^2 v_2}{dx_2^2} = -4x_2 \]

\[ EI \frac{dv_2}{dx_2} = -2x_2^2 + C_3 \quad (3) \]

\[ Elv_2 = -\frac{2}{3} x_2^3 + C_3 x_2 + C_4 \quad (4) \]
The four constants of integration are determined using three boundary conditions, namely, \( v_1 = 0 \) at \( x_1 = 0 \), \( v_1 = 0 \) at \( x_1 = 2 \) m, and \( v_2 = 0 \) at \( x_2 = 1 \) m, and one continuity equation. Here the continuity of slope at the roller requires \( dv_1/dx_1 = -dv_2/dx_2 \) at \( x_1 = 2 \) m and \( x_2 = 1 \) m. Why is there a negative sign in this equation? (Note that continuity of displacement at \( B \) has been indirectly considered in the boundary conditions, since \( v_1 = v_2 = 0 \) at \( x_1 = 2 \) m and \( x_2 = 1 \) m.) Applying these four conditions yields

\[
v_1 = 0 \text{ at } x_1 = 0; \quad 0 = 0 + 0 + C_2
\]

\[
v_1 = 0 \text{ at } x_1 = 2 \text{ m}; \quad 0 = \frac{-1}{3}(2)^3 + C_1(2) + C_2
\]
12. Deflections of Beams and Shafts

EXAMPLE 12.4 (SOLN)

Slope and Elastic curve:

\[ v_2 = 0 \text{ at } x_2 = 1 \text{ m}; \quad 0 = -\frac{2}{3}(1)^3 + C_3(1) + C_4 \]

\[ \frac{dv_1}{dx_1} \bigg|_{x = 2 \text{ m}} = \frac{dv_2}{dx_2} \bigg|_{x = 1 \text{ m}} ; \quad -(2)^2 + C_1 = -(-2(1)^2 + C_3) \]

Solving, we obtain

\[ C_1 = \frac{4}{3}, \quad C_2 = 0, \quad C_3 = \frac{14}{3}, \quad C_4 = -4 \]
12. Deflections of Beams and Shafts

EXAMPLE 12.4 (SOLN)

Slope and Elastic curve:

Substituting $C_3$ and $C_4$ into Eqn (4) gives

$$EIv_2 = -\frac{2}{3}x_2^3 + \frac{14}{3}x_2 - 4$$

Displacement at $C$ is determined by setting $x_2 = 0$,

$$v_C = -\frac{4\text{kN} \cdot \text{m}^3}{EI} \quad \text{Ans.}$$
If several different loadings act on the beam the method of integration becomes more tedious to apply, because separate loadings or moment functions must be written for each region of the beam.

8 constants of integration
12. Deflections of Beams and Shafts

*12.3 DISCONTINUITY FUNCTIONS

• A simplified method for finding the eqn of the elastic curve for a multiply loaded beam using a single expression, formulated from the loading on the beam, \( w = w(x) \), or the beam’s internal moment, \( M = M(x) \) is discussed below.

Discontinuity functions

Macaulay functions

• Such functions can be used to describe distributed loadings, written generally as

\[
\langle x - a \rangle^n = \begin{cases} 
0 & \text{for } x < a \\
(x - a)^n & \text{for } x \geq a 
\end{cases}
\]

\( n \geq 0 \)
Discontinuity functions

Macaulay functions

- $x$ represents the coordinate position of a point along the beam.
- $a$ is the location on the beam where a “discontinuity” occurs, or the point where a distributed loading begins.
- The functions describe both uniform load and triangular load.
12. Deflections of Beams and Shafts

*12.3 DISCONTINUITY FUNCTIONS

Discontinuity functions

Macaulay functions

<table>
<thead>
<tr>
<th>TABLE 12–2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loading</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>(1) $M_0$</td>
</tr>
<tr>
<td>(2) $P$</td>
</tr>
<tr>
<td>(3) $w_0$</td>
</tr>
<tr>
<td>(4) slope = $m$</td>
</tr>
</tbody>
</table>
Macaulay Functions. For purposes of beam or shaft deflection, Macaulay functions, named after the mathematician W. H. Macaulay, can be used to describe **distributed loadings**. These functions can be written in general form as

\[
\langle x - a \rangle^n = \begin{cases} 
0 & \text{for } x < a \\
(x - a)^n & \text{for } x \geq a \\
n \geq 0
\end{cases}
\] (12–11)

Here \( x \) represents the coordinate position of a point along the beam, and \( a \) is the location on the beam where a “discontinuity” occurs, namely the point where a distributed loading begins. Note that the Macaulay function \( \langle x - a \rangle^n \) is written with angle brackets to distinguish it from the ordinary function \( (x - a)^n \), written with parentheses. As stated by the equation, only when \( x \geq a \) is \( \langle x - a \rangle^n = (x - a)^n \), otherwise it is zero. Furthermore, these functions are valid only for exponential values \( n \geq 0 \). Integration of Macaulay functions follows the same rules as for ordinary functions, i.e.,

\[
\int \langle x - a \rangle^n \, dx = \frac{(x - a)^{n+1}}{n + 1} + C
\]

Note how the Macaulay functions describe both the uniform load \( w_0 (n = 0) \) and triangular load \( (n = 1) \), shown in Table 12–2, items 3 and 4. This type of description can, of course, be extended to distributed loadings having other forms. Also, it is possible to use superposition with
Singularity Functions. These functions are only used to describe the point location of concentrated forces or couple moments acting on a beam or shaft. Specifically, a concentrated force $P$ can be considered as a special case of a distributed loading, where the intensity of the loading is $w = P/\epsilon$ such that its length is $\epsilon$, where $\epsilon \to 0$, Fig. 12–15. The area under this loading diagram is equivalent to $P$, positive upward, and so we will use the singularity function

$$w = P(x - a)^{-1} = \begin{cases} 0 & \text{for } x \neq a \\ P & \text{for } x = a \end{cases}$$

(12–13)

to describe the force $P$. Here $n = -1$ so that the units for $w$ are force per length, as it should be. Furthermore, the function takes on the value of $P$ only at the point $x = a$ where the load occurs, otherwise it is zero.
12. Deflections of Beams and Shafts

Singularity functions

In a similar manner, a couple moment \( M_0 \), considered positive clockwise, is a limit as \( \epsilon \to 0 \) of two distributed loadings as shown in Fig. 12–16. Here the following function describes its value.

\[
w = M_0 (x - a)^{-2} = \begin{cases} 0 & \text{for } x \neq a \\ M_0 & \text{for } x = a \end{cases}
\]  

(12–14)

The exponent \( n = -2 \), in order to ensure that the units of \( w \), force per length, are maintained.

Application of Eqs. 12–11 through 12–15 provides a rather direct means for expressing the loading or the internal moment in a beam as a function of \( x \). When doing so, close attention must be paid to the signs of the external loadings. As stated above, and as shown in Table 12–2, concentrated forces and distributed loads are positive upward, and couple moments are positive clockwise. If this sign convention is followed, then the internal shear and moment are in accordance with the beam sign convention established in Sec. 6.1.
12. Deflections of Beams and Shafts

\[ w = 2.75 \text{kN} (x - 0)^{-1} + 1.5 \text{kN} \cdot \text{m}(x - 3 \text{ m})^{-2} - 3 \text{kN/m}(x - 3 \text{ m})^0 - 1 \text{kN/m}^2(x - 3 \text{ m})^1 \]

The reactive force at B is not included here since x is never greater than 6 m, and furthermore, this value is of no consequence in calculating the slope or deflection. We can determine the moment expression directly from Table 12–2, rather than integrating this expression twice. In either case,

\[ M = 2.75 \text{kN}(x - 0)^1 + 1.5 \text{kN} \cdot \text{m}(x - 3 \text{ m})^0 - \frac{3 \text{kN/m}}{2}(x - 3 \text{ m})^2 - \frac{1 \text{kN/m}^2}{6}(x - 3 \text{ m})^3 \]

\[ = 2.75x + 1.5(x - 3)^0 - 1.5(x - 3)^2 - \frac{1}{6}(x - 3)^3 \]

The deflection of the beam can now be determined after this equation is integrated two successive times and the constants of integration are evaluated using the boundary conditions of zero displacement at A and B.
**12. Deflections of Beams and Shafts**

*12.3 DISCONTINUITY FUNCTIONS*

Procedure for analysis

**Elastic curve**
- Sketch the beam’s elastic curve and identify the boundary conditions at the supports.
- Zero displacement occurs at all pin and roller supports, and zero slope and zero displacement occurs at fixed supports.
- Establish the $x$ axis so that it extends to the right and has its origin at the beam’s left end.

**Load or moment function**
- Calculate the support reactions and then use the discontinuity functions in Table 12-2 to express either the loading $w$ or the internal moment $M$ as a function of $x$. 
12. Deflections of Beams and Shafts

*12.3 DISCONTINUITY FUNCTIONS

Procedure for analysis

Load or moment function

- Calculate the support reactions and then use the discontinuity functions in Table 12-2 to express either the loading $w$ or the internal moment $M$ as a function of $x$.
- Make sure to follow the sign convention for each loading as it applies for this equation.
- Note that the distributed loadings must extend all the way to the beam’s right end to be valid. If this does not occur, use the method of superposition.
12. Deflections of Beams and Shafts

12.3 DISCONTINUITY FUNCTIONS

Procedure for analysis

Slope and elastic curve

- Substitute $w$ into $EI \frac{d^4 v}{dx^4} = -w(x)$ or $M$ into the moment curvature relation $EI \frac{d^2 v}{dx^2} = M$, and integrate to obtain the eqns for the beam’s slope and deflection.

- Evaluate the constants of integration using the boundary conditions, and substitute these constants into the slope and deflection eqns to obtain the final results.

- When the slope and deflection eqns are evaluated at any pt on the beam, a +ve slope is counterclockwise, and a +ve displacement is upward.
EXAMPLE 12.6

Determine the eqn of the elastic curve for the cantilevered beam shown. $EI$ is constant.
12. Deflections of Beams and Shafts

EXAMPLE 12.6 (SOLN)

Elastic curve

The loads cause the beam to deflect as shown. The boundary conditions require zero slope and displacement at A.
Loading functions

Support reactions shown on free-body diagram. Since distributed loading does not extend to $C$ as required, use superposition of loadings to represent same effect.
12. Deflections of Beams and Shafts

EXAMPLE 12.6 (SOLN)

Loading functions
Therefore, \( w = 52 \text{kN}(x - 0)^{-1} - 258 \text{kN} \cdot m(x - 0)^{-2} - 8 \text{kN/m}(x - 0)^0 \)
\[ + 50 \text{kN} \cdot m(x - 5 \text{ m})^{-2} + 8 \text{kN/m}(x - 5 \text{ m})^0 \]

The 12-kN load is not included, since \( x \) cannot be greater than 9 m. Because \( dV/dx = -w(x) \), then by integrating, neglect constant of integration since reactions are included in load function, we have

\[ V = 52(x - 0)^0 - 258(x - 0)^{-1} - 8(x - 0)^1 \]
\[ + 50(x - 5)^{-1} + 8(x - 5)^1 \]
Loading functions

Furthermore, \( dM/dx = V \), so integrating again yields

\[
M = -258(x - 0)^0 + 52(x - 0)^1 - \frac{1}{2}(8)(x - 0)^2 + 50(x - 5)^0 + \frac{1}{2}(8)(x - 5)^2
\]

\[
= \left( -258 + 52x - 4x^2 + 4(x - 5)^2 + 50(x - 5)^0 \right) \text{kN} \cdot \text{m}
\]

The same result can be obtained directly from Table 12-2.
Slope and elastic curve

Applying Eqn 12-10 and integrating twice, we have

\[ EI \frac{d^2 \nu}{dx^2} = -258 + 52x - 4x^2 + 50\langle x - 5 \rangle^0 + 4\langle x - 5 \rangle^2 \]

\[ EI \frac{d \nu}{dx} = -258x + 26x^2 - \frac{4}{3}x^3 + 50\langle x - 5 \rangle^1 + \frac{4}{3}\langle x - 5 \rangle^3 + C_1 \]

\[ EI \nu = -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x - 5 \rangle^2 + \frac{1}{3}\langle x - 5 \rangle^4 + C_1x + C_2 \]
12. Deflections of Beams and Shafts

EXAMPLE 12.6 (SOLN)

Slope and elastic curve

Since \( \frac{d\nu}{dx} = 0 \) at \( x = 0 \), \( C_1 = 0 \); and \( \nu = 0 \) at \( x = 0 \), so \( C_2 = 0 \). Thus

\[
\nu = \frac{1}{EI} \left( -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 
+ 25(x - 5)^2 + \frac{1}{3}(x - 5)^4 \right) \text{ m}
\]
Example 12.5

Determine the maximum deflection of the beam shown in Fig. 12–20a. $EI$ is constant.

(a)

(b)
Example 12.5 (SOLN)

**Loading Function.** The reactions have been calculated and are shown on the free-body diagram in Fig. 12–20b. The loading function for the beam can be written as

\[ w = 8 \text{kN}(x - 0)^{-1} - 6 \text{kN}(x - 10 \text{ m})^{-1} \]

The couple moment and force at B are not included here, since they are located at the right end of the beam, and \(x\) cannot be greater than 30 m. Applying \(dV/dx = -w(x)\), we get

\[ V = -8(x - 0)^0 + 6(x - 10)^0 \]

In a similar manner, \(dM/dx = V\) yields

\[ M = -8(x - 0)^1 + 6(x - 10)^1 \]

\[ = (-8x + 6(x - 10)^1) \text{kN} \cdot \text{m} \]

**Slope and Elastic Curve.** Integrating twice yields

\[ EI \frac{d^2\nu}{dx^2} = -8x + 6(x - 10)^1 \]

\[ EI \frac{d\nu}{dx} = -4x^2 + 3(x - 10)^2 + C_1 \]

\[ EI\nu = \frac{4}{3}x^3 + (x - 10)^3 + C_1x + C_2 \]
From Eq. 1, the boundary condition $v = 0$ at $x = 10$ m and $v = 0$ at $x = 30$ m gives

$$0 = -1333 + (10 - 10)^3 + C_1(10) + C_2$$
$$0 = -36000 + (30 - 10)^3 + C_1(30) + C_2$$

Solving these equations simultaneously for $C_1$ and $C_2$, we get $C_1 = 1333$ and $C_2 = -12000$. Thus,

$$EI \frac{dv}{dx} = -4x^2 + 3(x - 10)^2 + 1333 \quad (2)$$

$$EIv = -\frac{4}{3}x^3 + (x - 10)^3 + 1333x - 12000 \quad (3)$$

From Fig. 12-20a, maximum displacement may occur either at $C$, or at $D$, where the slope $dv/dx = 0$. To obtain the displacement of $C$, set $x = 0$ in Eq. 3. We get

$$v_C = -\frac{12000 \text{ kN} \cdot \text{m}^3}{EI}$$
To locate point $D$, use Eq. 2 with $x > 10$ m and $dv/dx = 0$.

\[ 0 = -4x_D^2 + 3(x_D - 10)^2 + 1333 \]
\[ x_D^2 + 60x_D - 1633 = 0 \]

Solving for the positive root,

\[ x_D = 20.3 \text{ m} \]

Hence, from Eq. 3,

\[ EIv_D = -\frac{4}{3}(20.3)^3 + (20.3 - 10)^3 + 1333(20.3) - 12000 \]

\[ v_D = \frac{5000 \text{ kN} \cdot \text{m}^3}{EI} \]

Comparing this value with $v_C$, we see that $v_{\text{max}} = v_C$. 

\textit{Ans.}
12. Deflections of Beams and Shafts

12.5 METHOD OF SUPERPOSITION

- The differential eqn $EI \frac{d^4 v}{dx^4} = -w(x)$ satisfies the two necessary requirements for applying the principle of superposition.

- The load $w(x)$ is linearly related to the deflection $v(x)$.

- The load is assumed not to change significantly the original geometry of the beam or shaft.

- The following examples explain this method.
Example 12.3

Determine the displacement at point C and the slope at the support A of the beam shown in Fig. 12–29a. $EI$ is constant.

**Fig. 12–29**

**Solution**
Example 12.3 (SOLN)

Solution

The loading can be separated into two component parts as shown in Figs. 12–29b and 12–29c. The displacement at C and slope at A are found using the table in Appendix C for each part.

For the distributed loading,

\[
(\theta_A)_1 = \frac{3wL^3}{128EI} = \frac{3(2 \text{kN/m})(8 \text{ m})^3}{128EI} = \frac{24 \text{kN} \cdot \text{m}^2}{EI}
\]

\[
(v_C)_1 = \frac{5wL^4}{768EI} = \frac{5(2 \text{kN/m})(8 \text{ m})^4}{768EI} = \frac{53.33 \text{kN} \cdot \text{m}^3}{EI}
\]

For the 8-kN concentrated force,

\[
(\theta_A)_2 = \frac{PL^2}{16EI} = \frac{8 \text{kN}(8 \text{ m})^2}{16EI} = \frac{32 \text{kN} \cdot \text{m}^2}{EI}
\]

\[
(v_C)_2 = \frac{PL^3}{48EI} = \frac{8 \text{kN}(8 \text{ m})^3}{48EI} = \frac{85.33 \text{kN} \cdot \text{m}^3}{EI}
\]

The total displacement at C and the slope at A are the algebraic sums of these components. Hence,

\[
(\theta_A) = (\theta_A)_1 + (\theta_A)_2 = \frac{56 \text{kN} \cdot \text{m}^2}{EI} \quad \text{Ans.}
\]

\[
(v_C) = (v_C)_1 + (v_C)_2 = \frac{139 \text{kN} \cdot \text{m}^3}{EI} \quad \text{Ans.}
\]
EXAMPLE 12.14

Determine the displacement at the end C of the overhanging beam shown in Fig. 12–29a. EI is constant.

SOLUTION

Since the table in Appendix C does not include beams with overhangs, the beam will be separated into a simply supported and a cantilevered portion. First we will calculate the slope at B, as caused by the distributed load acting on the simply supported span, Fig. 12–29b.

\[
(\theta_B)_1 = \frac{wL^3}{24EI} = \frac{5 \text{ kN/m}(4 \text{ m})^3}{24EI} = \frac{13.33 \text{ kN} \cdot \text{m}^2}{EI}
\]

Since this angle is small, \((\theta_B)_1 \approx \tan(\theta_B)_1\), and the vertical displacement at point C is

\[
(v_C)_1 = (2 \text{ m}) \left( \frac{13.33 \text{ kN} \cdot \text{m}^2}{EI} \right) = \frac{26.67 \text{ kN} \cdot \text{m}^3}{EI}
\]

\[
(\theta_B)_2 = \frac{M_0L}{3EI} = \frac{20 \text{ kN} \cdot \text{m}(4 \text{ m})}{3EI} = \frac{26.67 \text{ kN} \cdot \text{m}^2}{EI}
\]

so that the extended point C is displaced

\[
(v_C)_2 = (2 \text{ m}) \left( \frac{26.7 \text{ kN} \cdot \text{m}^2}{EI} \right) = \frac{53.33 \text{ kN} \cdot \text{m}^3}{EI}
\]

Finally, the cantilevered portion BC is displaced by the 10-kN force, Fig. 12–29d. We have

\[
(v_C)_3 = \frac{PL^3}{3EI} = \frac{10 \text{ kN}(2 \text{ m})^3}{3EI} = \frac{26.67 \text{ kN} \cdot \text{m}^3}{EI}
\]

Summing these results algebraically, we obtain the displacement of point C,

\[
v_C = -\frac{26.7}{EI} + \frac{53.3}{EI} + \frac{26.7}{EI} = \frac{53.3 \text{ kN} \cdot \text{m}^3}{EI}
\]

\[\text{Ans.}\]
EXAMPLE 12.16

Steel bar shown is supported by two springs at its ends $A$ and $B$. Each spring has a stiffness $k = 45 \text{ kN/m}$ and is originally unstretched. If the bar is loaded with a force of 3 kN at pt $C$, determine the vertical displacement of the force. Neglect the weight of the bar and take $E_{st} = 200 \text{ GPa}$, $I = 4.6875 \times 10^{-6} \text{ m}$. 
End reactions at $A$ and $B$ are computed and shown. Each spring deflects by an amount

$$\left(\nu_A\right)_1 = \frac{2 \text{ kN}}{45 \text{ kN/m}} = 0.0444 \text{ m}$$

$$\left(\nu_B\right)_1 = \frac{1 \text{ kN}}{45 \text{ kN/m}} = 0.0222 \text{ m}$$
EXAMPLE 12.16 (SOLN)

If bar is considered rigid, these displacements cause it to move into positions shown. For this case, the vertical displacement at \( C \) is

\[
(\psi_C)_1 = (\psi_B)_1 + \frac{2}{3} \frac{m}{m}[ (\psi_A)_1 - (\psi_B)_1 ]
\]

\[
= 0.0222 \text{ m} + \frac{2}{3} \left[ 0.0444 \text{ m} - 0.0282 \text{ m} \right]
\]

\[
= 0.0370 \text{ m}
\]
12. Deflections of Beams and Shafts

**EXAMPLE 12.16 (SOLN)**

We can find the displacement at \( C \) caused by the deformation of the bar, by using the table in Appendix C. We have

\[
(v_C) = \frac{Pab}{6EIL} \left( L^2 - b^2 - a^2 \right)
\]

\[
= \frac{(3 \text{ kN})(1 \text{ m})(2 \text{ m})\left[(3 \text{ m})^2 - (2 \text{ m})^2 - (1 \text{ m})^2\right]}{6(200)(10^6) \text{kN/m}^2(4.6875)\left(10^{-6}\right) \text{m}^4(3 \text{ m})}
\]

\[
= 1.422 \text{ mm}
\]
Adding the two displacement components, we get

\[ (+\downarrow) \quad \nu_C = 0.0370 \text{ m} + 0.001422 \text{ m} \]

\[ = 0.0384 \text{ m} = 38.4 \text{ mm} \]