*12.4 SLOPE & DISPLACEMENT BY THE MOMENT-AREA METHOD

- Assumptions:
  - beam is initially straight,
  - is elastically deformed by the loads, such that the slope and deflection of the elastic curve are very small, and
  - deformations are caused by bending.

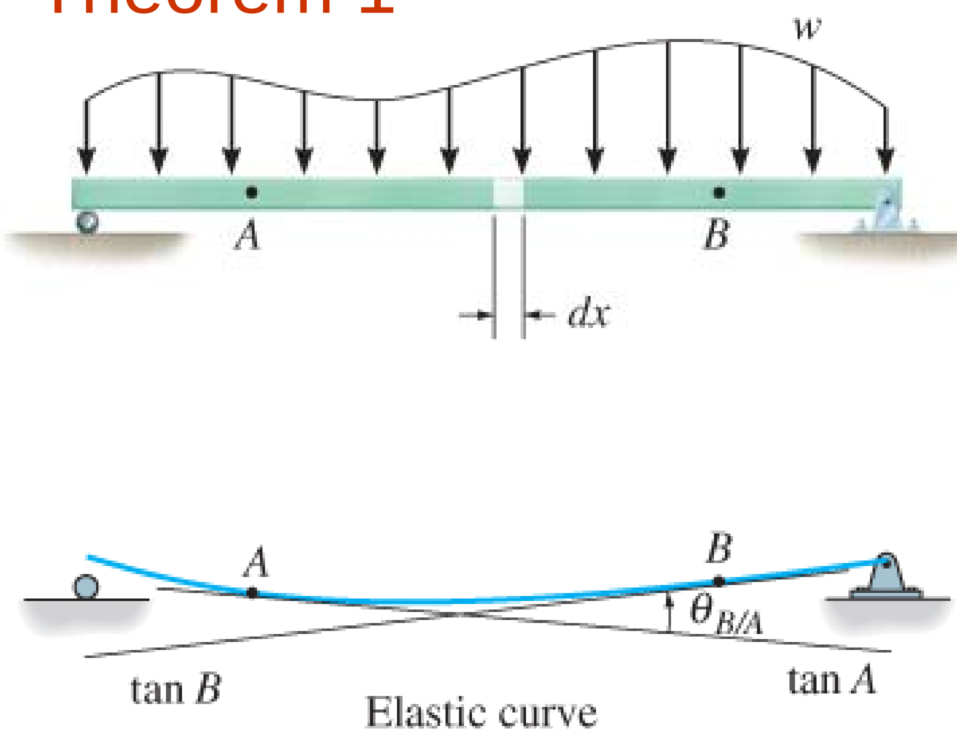
Theorem 1

- The angle between the tangents at any two pts on the elastic curve equals the area under the *M/EI* diagram between these two pts.

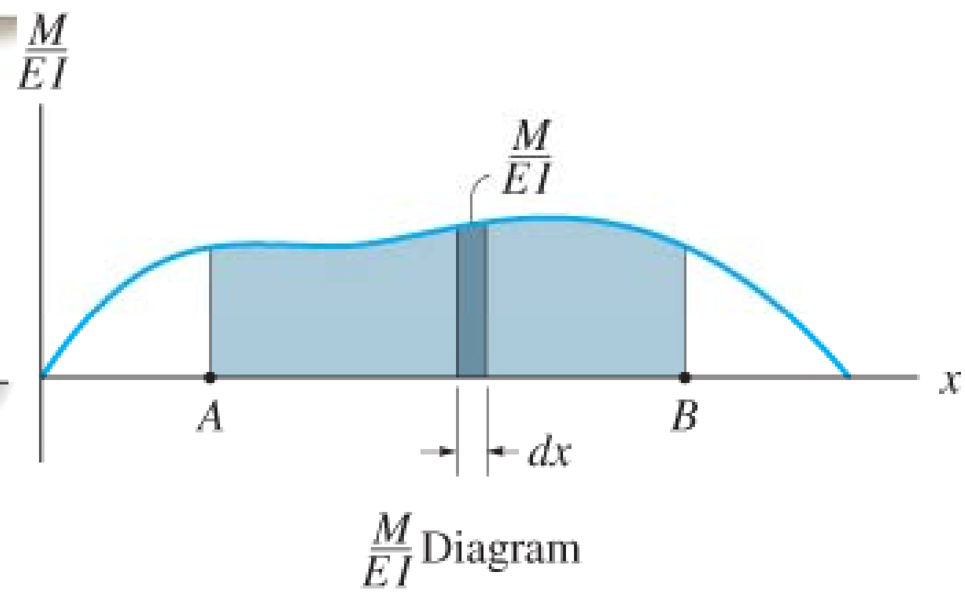
$$\theta_{B/A} = \int_A^B \frac{M}{EI}\,dx \qquad (12\text{-}19)$$

*12.4 SLOPE & DISPLACEMENT BY THE MOMENT-AREA METHOD

Theorem 1



(a)



(c)

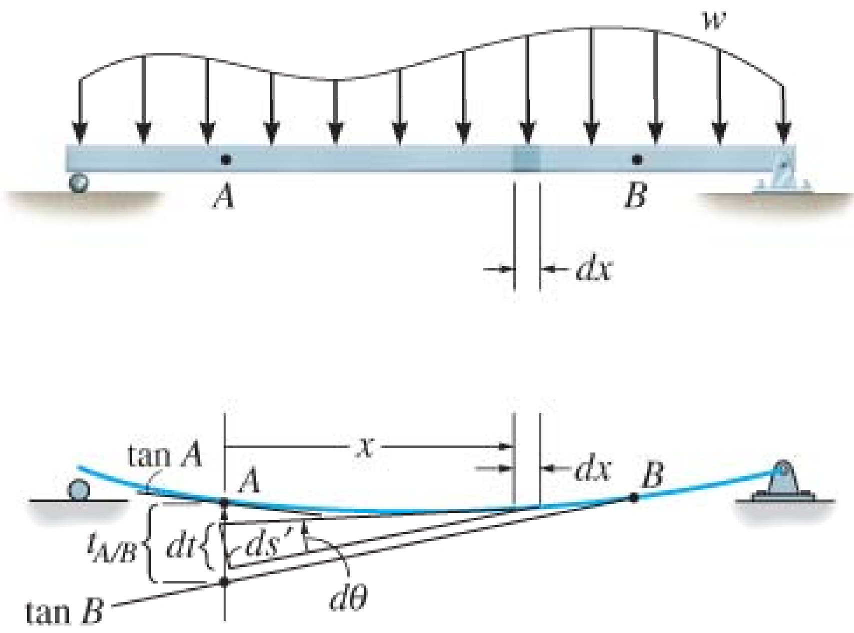
*12.4 SLOPE & DISPLACEMENT BY THE MOMENT-AREA METHOD

Theorem 2

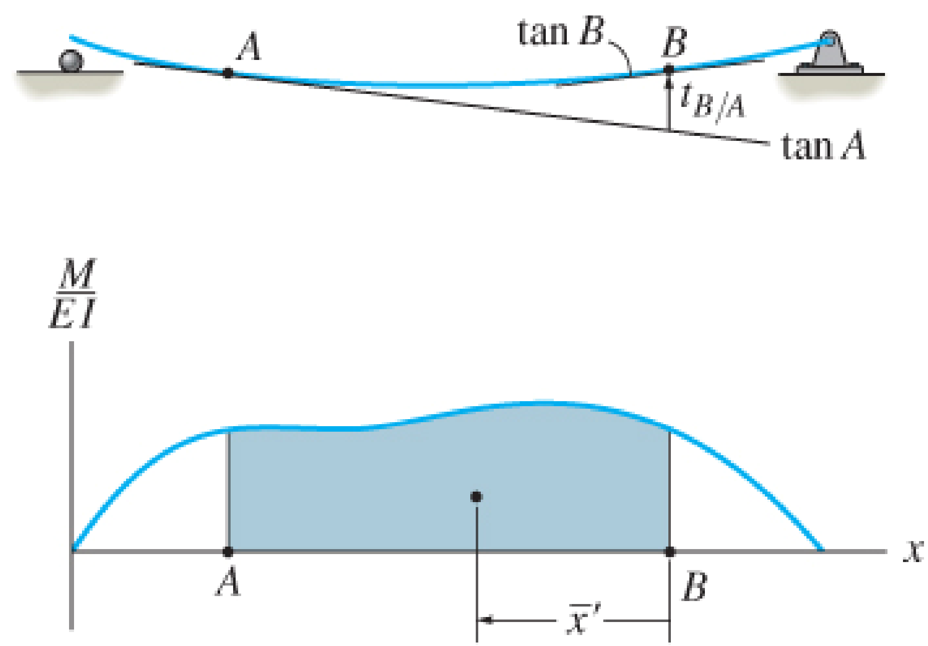
- The vertical deviation of the tangent at a pt (A) on the elastic curve w.r.t. the tangent extended from another pt (B) equals the moment of the area under the ME/I diagram between these two pts (A and B).
- This moment is computed about pt (A) where the vertical deviation ($t_{A/B}$) is to be determined.

*12.4 SLOPE & DISPLACEMENT BY THE MOMENT-AREA METHOD

Theorem 2



(a)



(c)

*12.4 SLOPE & DISPLACEMENT BY THE MOMENT-AREA METHOD

Procedure for analysis

M/EI Diagram

- Determine the support reactions and draw the beam's *M/EI* diagram.
- If the beam is loaded with concentrated forces, the *M/EI* diagram will consist of a series of straight line segments, and the areas and their moments required for the moment-area theorems will be relatively easy to compute.
- If the loading consists of a series of distributed loads, the *M/EI* diagram will consist of parabolic or perhaps higher-order curves, and we use the table on the inside front cover to locate the area and centroid under each curve.

*12.4 SLOPE & DISPLACEMENT BY THE MOMENT-AREA METHOD

Procedure for analysis

Elastic curve

- Draw an exaggerated view of the beam's elastic curve.
- Recall that pts of zero slope and zero displacement always occur at a fixed support, and zero displacement occurs at all pin and roller supports.
- If it is difficult to draw the general shape of the elastic curve, use the moment (M/EI) diagram.
- Realize that when the beam is subjected to a +ve moment, the beam bends concave up, whereas -ve moment bends the beam concave down.

*12.4 SLOPE & DISPLACEMENT BY THE MOMENT-AREA METHOD

Procedure for analysis

Elastic curve

- An inflection pt or change in curvature occurs when the moment if the beam (or M/EI) is zero.
- The unknown displacement and slope to be determined should be indicated on the curve.
- Since moment-area theorems apply only between two tangents, attention should be given as to which tangents should be constructed so that the angles or deviations between them will lead to the solution of the problem.
- The tangents at the supports should be considered, since the beam usually has zero displacement and/or zero slope at the supports.

*12.4 SLOPE & DISPLACEMENT BY THE MOMENT-AREA METHOD

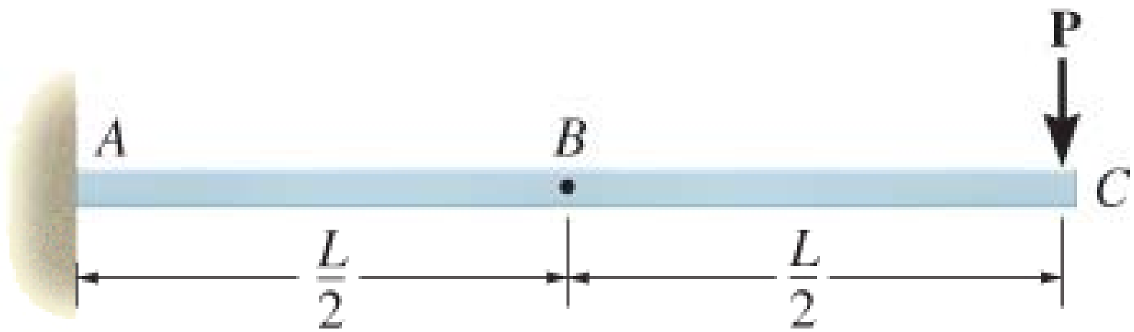
Procedure for analysis

Moment-area theorems

- Apply Theorem 1 to determine the angle between any two tangents on the elastic curve and Theorem 2 to determine the tangential deviation.
- The algebraic sign of the answer can be checked from the angle or deviation indicated on the elastic curve.
- A positive $\theta_{B/A}$ represents a counterclockwise rotation of the tangent at B w.r.t. tangent at A, and a +ve $t_{B/A}$ indicates that pt B on the elastic curve lies above the extended tangent from pt A.

EXAMPLE 12.7

Determine the slope of the beam shown at pts B and C. EI is constant.



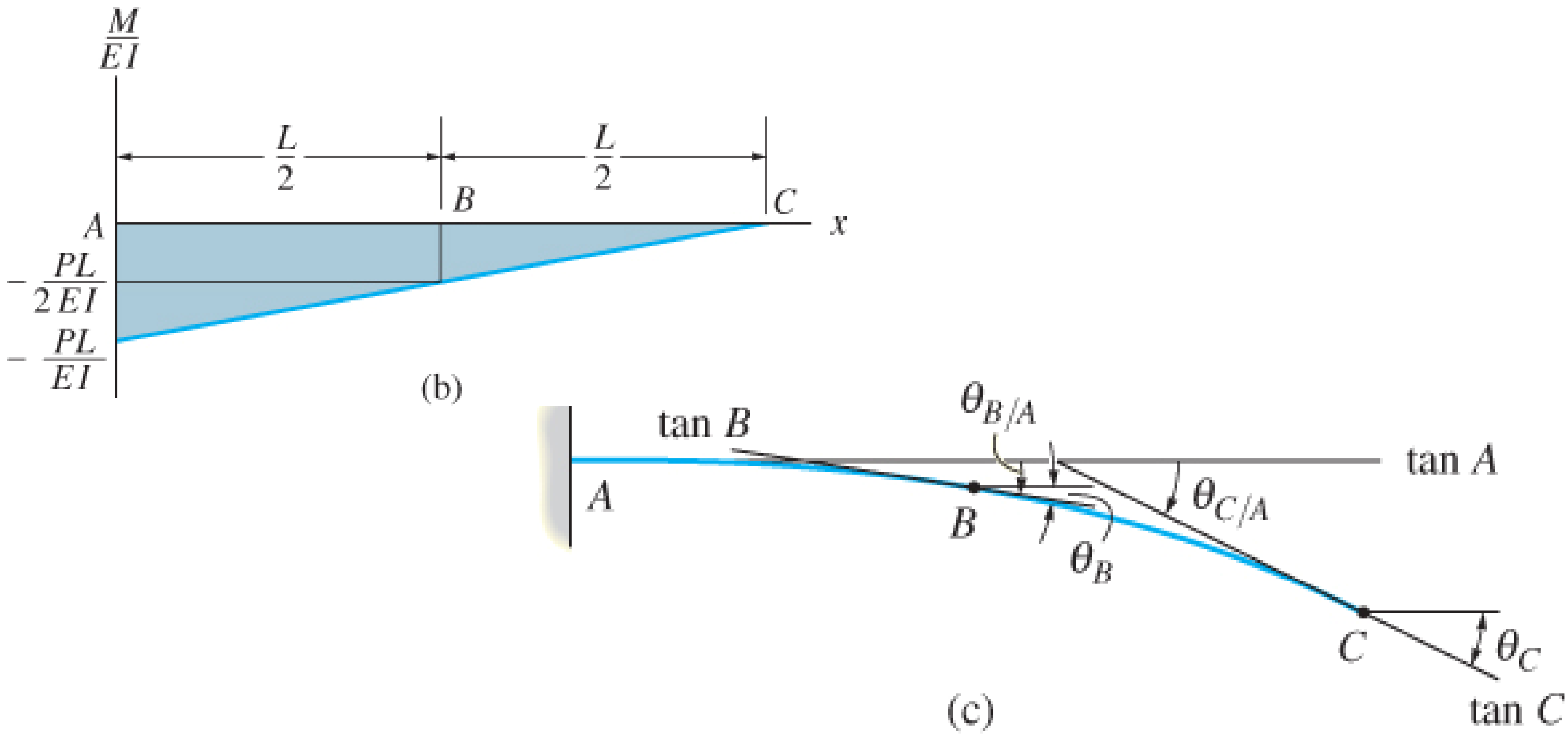
(a)

EXAMPLE 12.7 (SOLN)

M/EI diagram: See below.

Elastic curve:

The force **P** causes the beam to deflect as shown.



(b)

(c)

EXAMPLE 12.7 (SOLN)

Elastic curve:

The tangents at B and C are indicated since we are required to find B and C. Also, the tangent at the support (A) is shown. This tangent has a known zero slope. By construction, the angle between tan A and tan B, $\theta_{B/A}$, is equivalent to θ_B, or

$$\theta_B = \theta_{B/A} \quad \text{and} \quad \theta_C = \theta_{C/A}$$

EXAMPLE 12.7 (SOLN)

Moment-area theorem:

Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the M/EI diagram between pts A and B, that is,

$$\theta_B = \theta_{B/A} = \left(-\frac{PL}{2EI}\right)\left(\frac{L}{2}\right) + \frac{1}{2}\left(-\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)$$

$$= -\frac{3PL^2}{8EI}$$

EXAMPLE 12.7 (SOLN)

Moment-area theorem:

The negative sign indicates that angle measured from tangent at A to tangent at B is clockwise. This checks, since beam slopes downward at B.

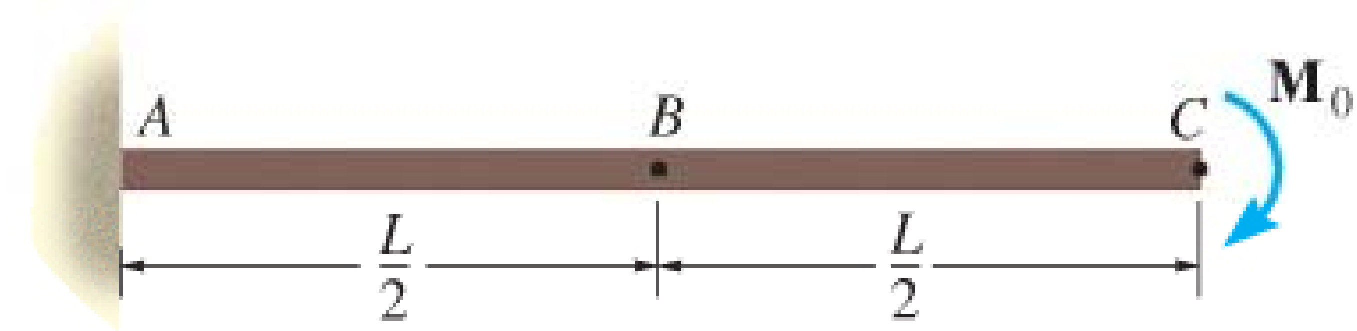
Similarly, area under the M/EI diagram between pts A and C equals $\theta_{C/A}$. We have

$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{PL}{EI}\right)L$$

$$= -\frac{PL^2}{2EI}$$

EXAMPLE 12.8

Determine the displacement of pts B and C of beam shown. EI is constant.



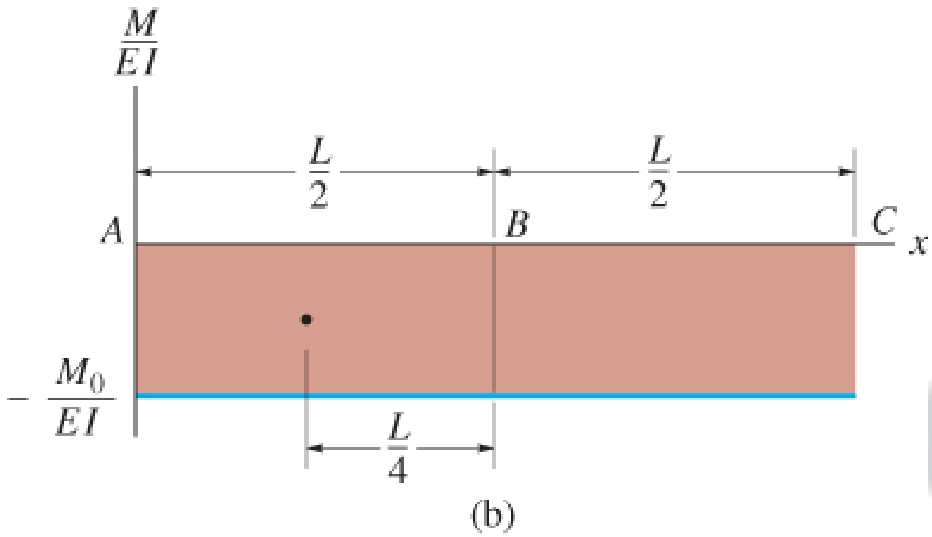
(a)

EXAMPLE 12.8 (SOLN)

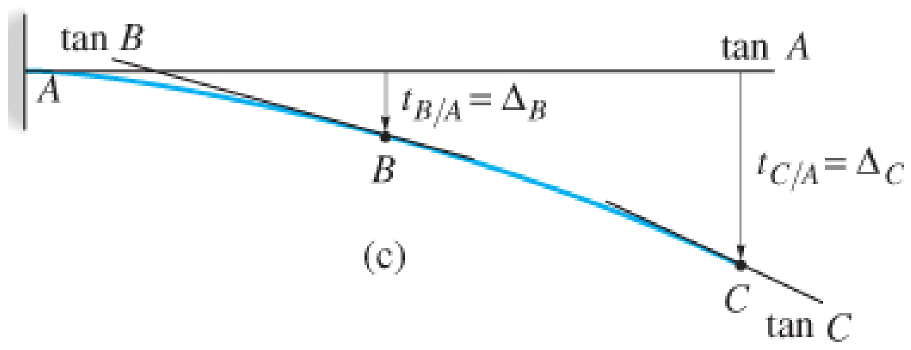
M/EI diagram: See below.

Elastic curve:

The couple moment at C cause the beam to deflect as shown.



(b)



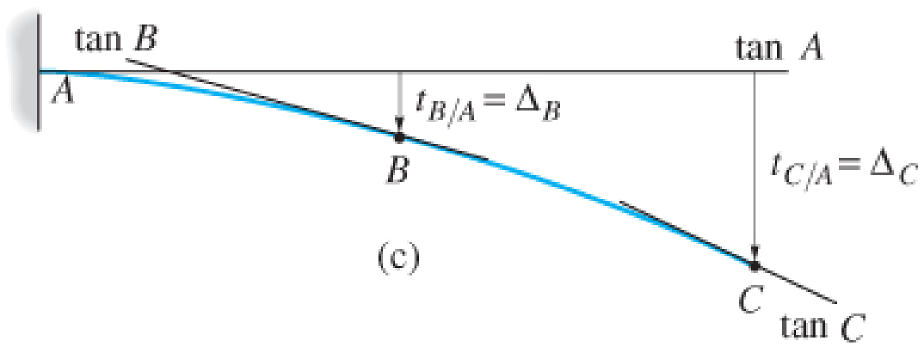
(c)

EXAMPLE 12.8 (SOLN)

Elastic curve:

The required displacements can be related directly to deviations between the tangents at B and A and C and A. Specifically, Δ_B is equal to deviation of tan A from tan B,

$$\Delta_B = t_{B/A} \qquad \Delta_C = t_{C/A}$$



(c)

EXAMPLE 12.8 (SOLN)

Moment-area theorem:

Applying Theorem 2, $t_{B/A}$ is equal to the moment of the shaded area under the M/EI diagram between A and B computed about pt B, since this is the pt where tangential deviation is to be determined. Hence,

$$\Delta_B = t_{B/A} = \left(\frac{L}{4}\right)\left[\left(-\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\right] = -\frac{M_0 L^2}{8EI}$$

EXAMPLE 12.8 (SOLN)

Moment-area theorem:

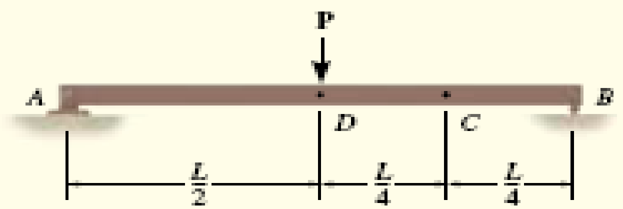
Likewise, for $t_{C/A}$ we must determine the moment of the area under the entire M/EI diagram from A to C about pt C. We have

$$\Delta_C = t_{C/A} = \left(\frac{L}{2}\right)\left[\left(-\frac{M_0}{EI}\right)(L)\right] = -\frac{M_0 L^2}{2EI}$$

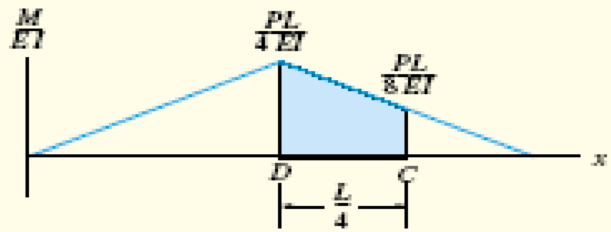
Since both answers are –ve, they indicate that pts B and C lie below the tangent at A. This checks with the figure.

EXAMPLE 12.9

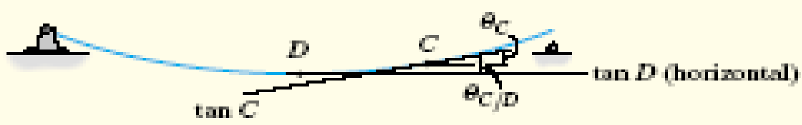
Determine the slope at point C of the beam in Fig. 12–25a. EI is constant.



(a)



(b)



(c)

$$\theta_C = \theta_{C/D}$$

Moment-Area Theorem. Using Theorem 1, $\theta_{C/D}$ is equal to the shaded area under the M/EI diagram between points D and C. We have

$$\theta_C = \theta_{C/D} = \left(\frac{PL}{8EI}\right)\left(\frac{L}{4}\right) + \frac{1}{2}\left(\frac{PL}{4EI} - \frac{PL}{8EI}\right)\left(\frac{L}{4}\right) = \frac{3PL^2}{64EI} \quad \textit{Ans.}$$

EXAMPLE 12.10

Determine the slope at point C for the steel beam in Fig. 12–26a. Take $E_{st} = 200$ GPa, $I = 17(10^6)$ mm^4.

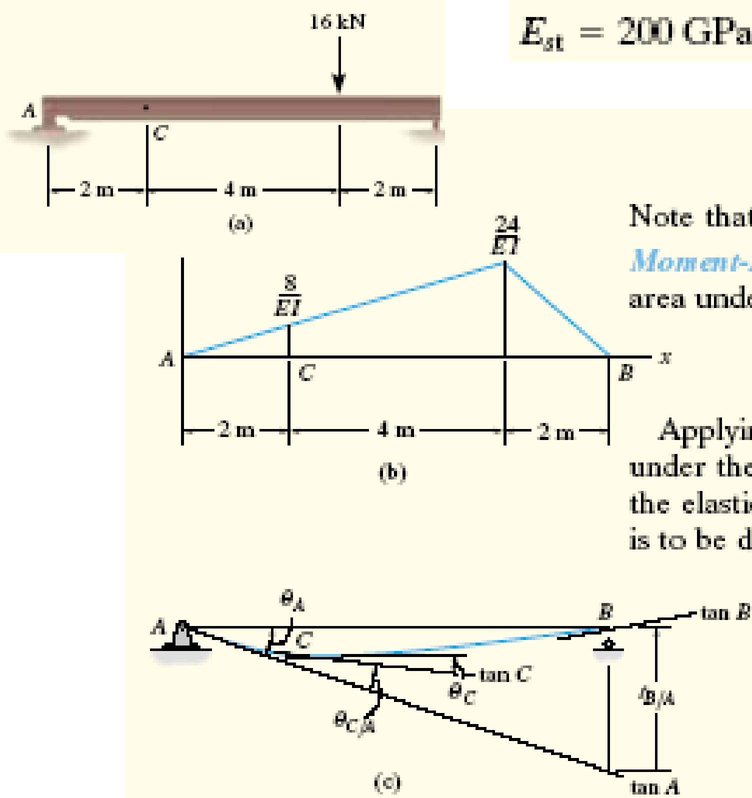


Fig. 12–26

$$|\theta_C| = |\theta_A| - |\theta_{C/A}| = \left|\frac{t_{B/A}}{8}\right| - |\theta_{C/A}| \qquad (1)$$

Note that Example 12.9 could also be solved using this method.

Moment-Area Theorems. Using Theorem 1, $\theta_{C/A}$ is equivalent to the area under the M/EI diagram between points A and C; that is,

$$\theta_{C/A} = \frac{1}{2}(2\text{ m})\left(\frac{8\text{ kN}\cdot\text{m}}{EI}\right) = \frac{8\text{ kN}\cdot\text{m}^2}{EI}$$

Applying Theorem 2, $t_{B/A}$ is equivalent to the moment of the area under the M/EI diagram between B and A about point B (the point on the elastic curve), since this is the point where the tangential deviation is to be determined. We have

$$t_{B/A} = \left(2\text{ m} + \frac{1}{3}(6\text{ m})\right)\left[\frac{1}{2}(6\text{ m})\left(\frac{24\text{ kN}\cdot\text{m}}{EI}\right)\right] + \left(\frac{2}{3}(2\text{ m})\right)\left[\frac{1}{2}(2\text{ m})\left(\frac{24\text{ kN}\cdot\text{m}}{EI}\right)\right] = \frac{320\text{ kN}\cdot\text{m}^3}{EI}$$

Substituting these results into Eq. 1, we get

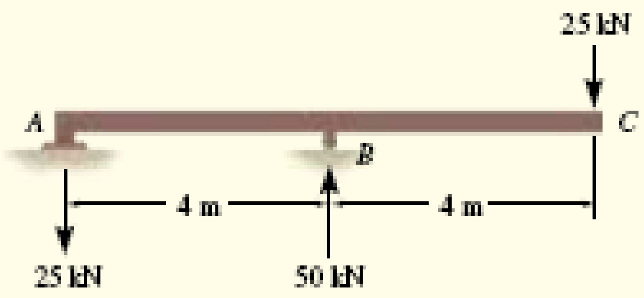
$$\theta_C = \frac{320\text{ kN}\cdot\text{m}^2}{(8\text{m})EI} - \frac{8\text{ kN}\cdot\text{m}^2}{EI} = \frac{32\text{ kN}\cdot\text{m}^2}{EI} \circlearrowright$$

We have calculated this result in units of kN and m, so converting EI into these units, we have

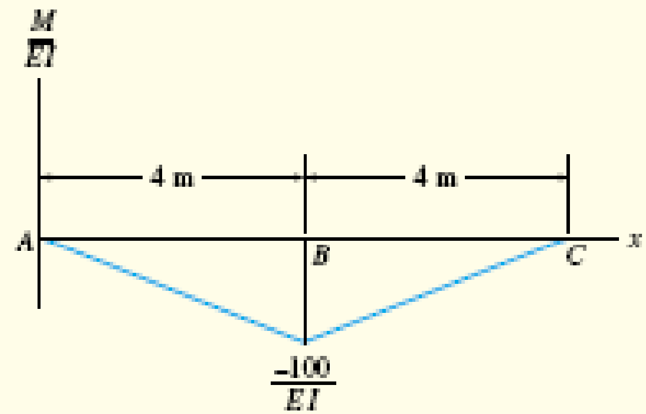
$$\theta_C = \frac{32\text{ kN}\cdot\text{m}^2}{200(10^6)\text{ kN/m}^2\,17(10^{-6})\text{ m}^4} = 0.00941\text{ rad} \circlearrowright \qquad \textit{Ans.}$$

EXAMPLE 12.12

Determine the displacement at point C for the steel overhanging beam shown in Fig. 12–28*a*. Take $E_{st} = 200$ GPa, $I = 50 \times 10^6$ mm^4.



(a)



(b)

Solution

M/EI Diagram. See Fig. 12–28*b*.

Elastic Curve. The loading causes the beam to deflect as shown in Fig. 12–28*c*. We are required to find Δ_C. By constructing tangents at C and at the supports A and B, it is seen that $\Delta_C = |t_{C/A}| - \Delta'$. However, Δ' can be related to $t_{B/A}$ by proportional triangles; that is, $\Delta'/8 = |t_{B/A}|/4$ or $\Delta' = 2|t_{B/A}|$. Hence,

$$\Delta_C = |t_{C/A}| - 2|t_{B/A}|$$

Moment-Area Theorem. Applying Theorem 2 to determine $t_{C/A}$ and $t_{B/A}$, we have

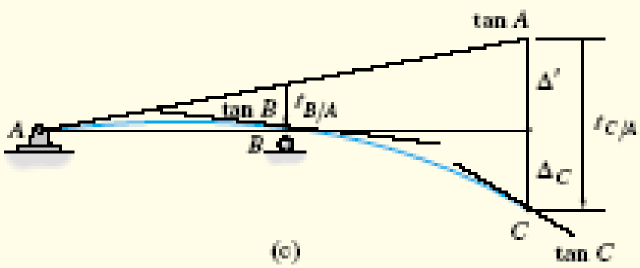


Fig. 12–28

$$t_{C/A} = (4\text{ m})\left(\frac{1}{2}(8\text{ m})\left(-\frac{100\text{ kN}\cdot\text{m}}{EI}\right)\right)$$

$$= -\frac{1600\text{ kN}\cdot\text{m}^3}{EI}$$

$$t_{B/A} = \left(\frac{1}{3}(4\text{ m})\right)\left[\frac{1}{2}(4\text{ m})\left(-\frac{100\text{ kN}\cdot\text{m}}{EI}\right)\right] = -\frac{266.67\text{ kN}\cdot\text{m}^3}{EI}$$

Why are these terms negative? Substituting the results into Eq. 1 yields

$$\Delta_C = \frac{1600\text{ kN}\cdot\text{m}^3}{EI} - 2\left(\frac{266.67\text{ kN}\cdot\text{m}^3}{EI}\right) = \frac{1066.66\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

Realizing that the computations were made in units of kN and m, we have

$$\Delta_C = \frac{1066.66\text{ kN}\cdot\text{m}^3(10^3\text{ mm/m})^3}{(200\text{ kN/mm}^2)[50(10^6)\text{ mm}^4]} = 106.7\text{ mm}\downarrow \qquad \textit{Ans.}$$

12.5 METHOD OF SUPERPOSITION

- The differential eqn $EI\, d^4v/dx^4 = -w(x)$ satisfies the two necessary requirements for applying the principle of superposition
- The load $w(x)$ is linearly related to the deflection $v(x)$
- The load is assumed not to change significantly the original geometry of the beam or shaft.

EXAMPLE 12.13

Determine the displacement at point C and the slope at the support A of the beam shown in Fig. 12–29a. EI is constant.

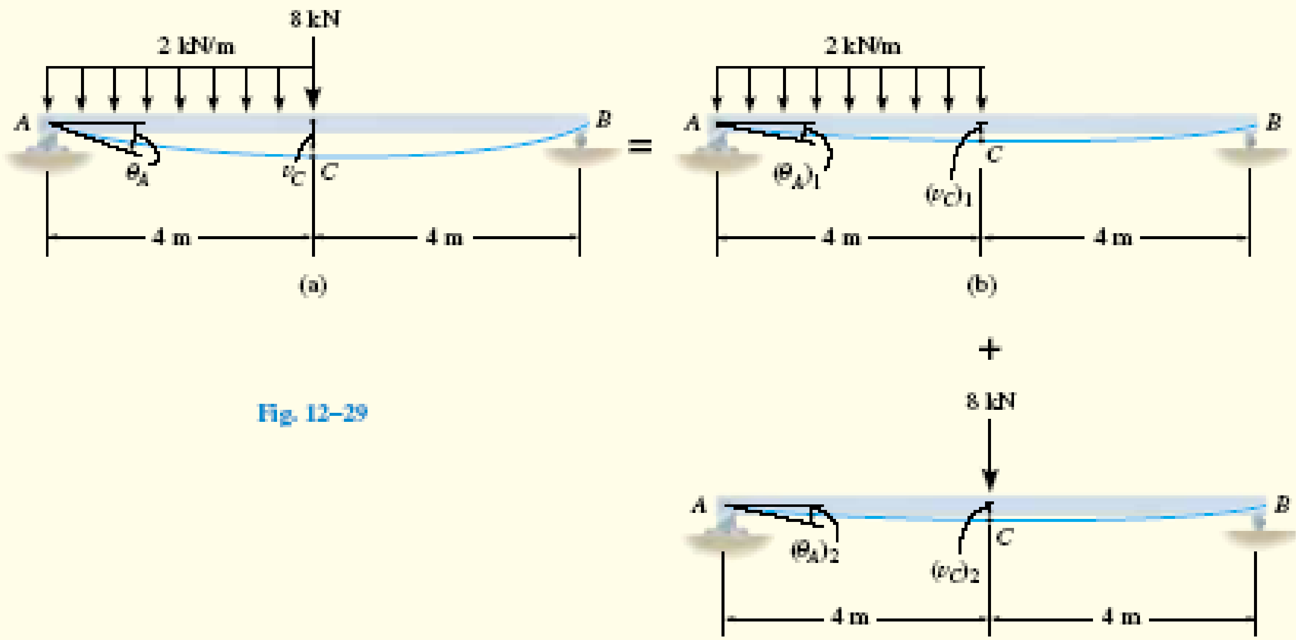


Fig. 12–29

Solution

The loading can be separated into two component parts as shown in Figs. 12–29*b* and 12–29*c*. The displacement at *C* and slope at *A* are found using the table in Appendix C for each part.

For the distributed loading,

$$(\theta_A)_1 = \frac{3wL^3}{128EI} = \frac{3(2\text{ kN/m})(8\text{ m})^3}{128EI} = \frac{24\text{ kN}\cdot\text{m}^2}{EI}\downarrow$$

$$(v_C)_1 = \frac{5wL^4}{768EI} = \frac{5(2\text{ kN/m})(8\text{ m})^4}{768EI} = \frac{53.33\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

For the 8-kN concentrated force,

$$(\theta_A)_2 = \frac{PL^2}{16EI} = \frac{8\text{ kN}(8\text{ m})^2}{16EI} = \frac{32\text{ kN}\cdot\text{m}^2}{EI}\downarrow$$

$$(v_C)_2 = \frac{PL^3}{48EI} = \frac{8\text{ kN}(8\text{ m})^3}{48EI} = \frac{85.33\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

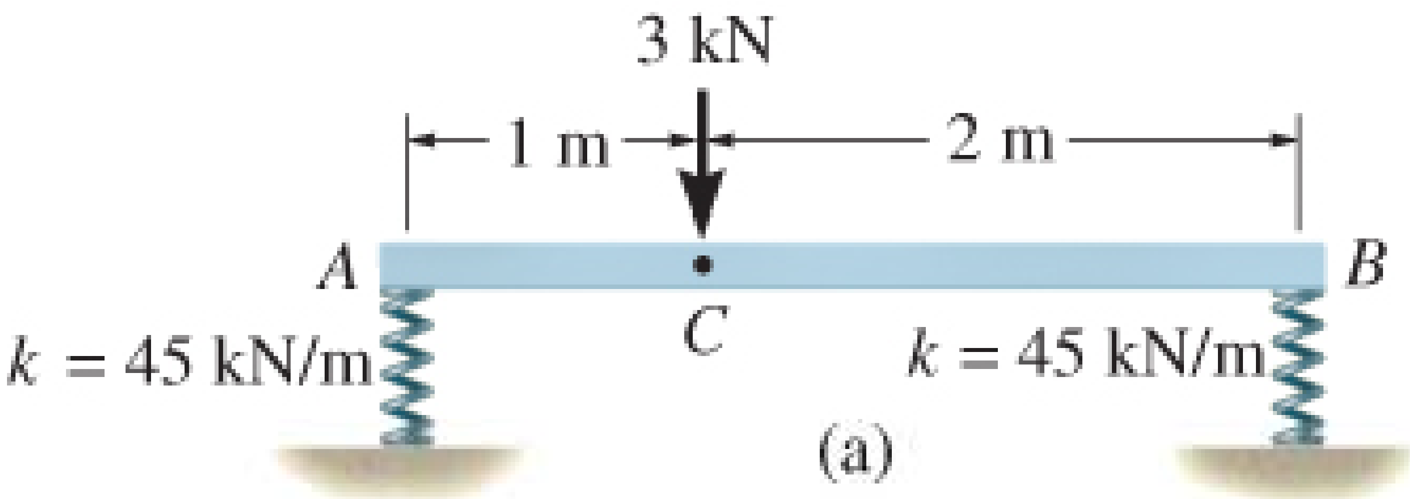
The total displacement at *C* and the slope at *A* are the algebraic sums of these components. Hence,

$$(+\downarrow) \qquad \theta_A = (\theta_A)_1 + (\theta_A)_2 = \frac{56\text{ kN}\cdot\text{m}^2}{EI}\downarrow \qquad \textit{Ans.}$$

$$(+\downarrow) \qquad v_C = (v_C)_1 + (v_C)_2 = \frac{139\text{ kN}\cdot\text{m}^3}{EI}\downarrow \qquad \textit{Ans.}$$

EXAMPLE 12.16

Steel bar shown is supported by two springs at its ends A and B. Each spring has a stiffness k = 45 kN/m and is originally unstretched. If the bar is loaded with a force of 3 kN at pt C, determine the vertical displacement of the force. Neglect the weight of the bar and take E_{st} = 200 GPa, I = 4.6875×10^{-6} m.



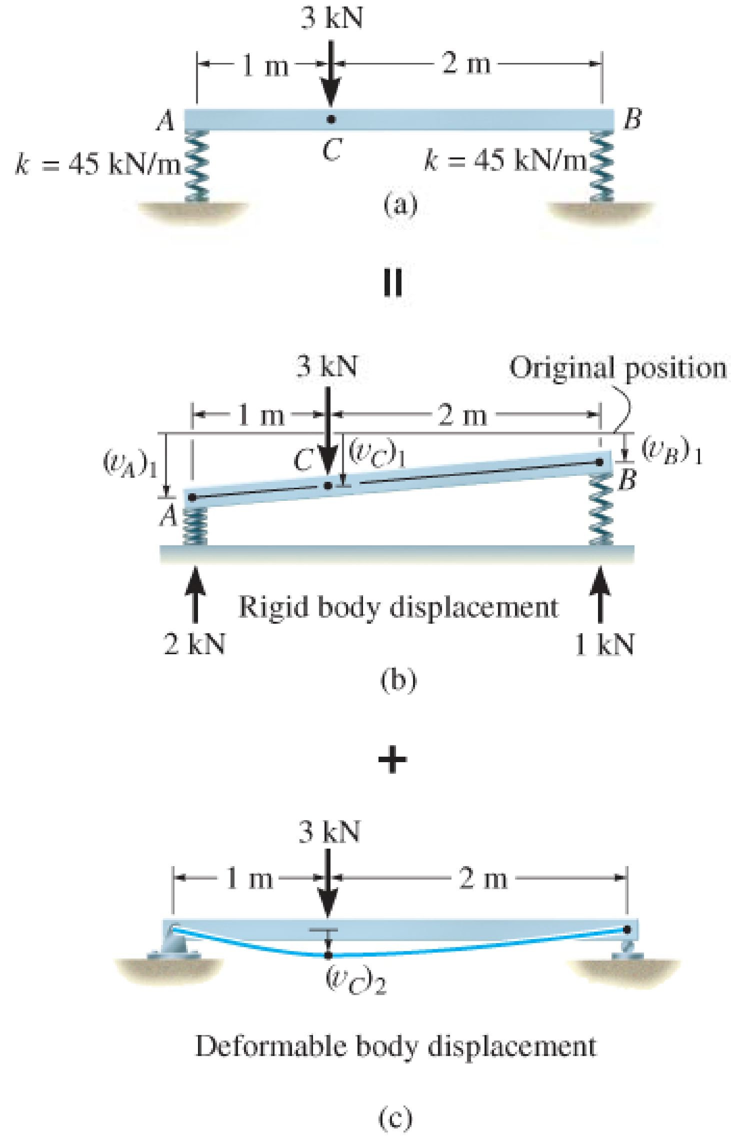
(a)

EXAMPLE 12.16 (SOLN)

End reactions at A and B are computed and shown. Each spring deflects by an amount

$$(\upsilon_A)_1 = \frac{2\ \text{kN}}{45\ \text{kN/m}} = 0.0444\ \text{m}$$

$$(\upsilon_B)_1 = \frac{1\ \text{kN}}{45\ \text{kN/m}} = 0.0222\ \text{m}$$



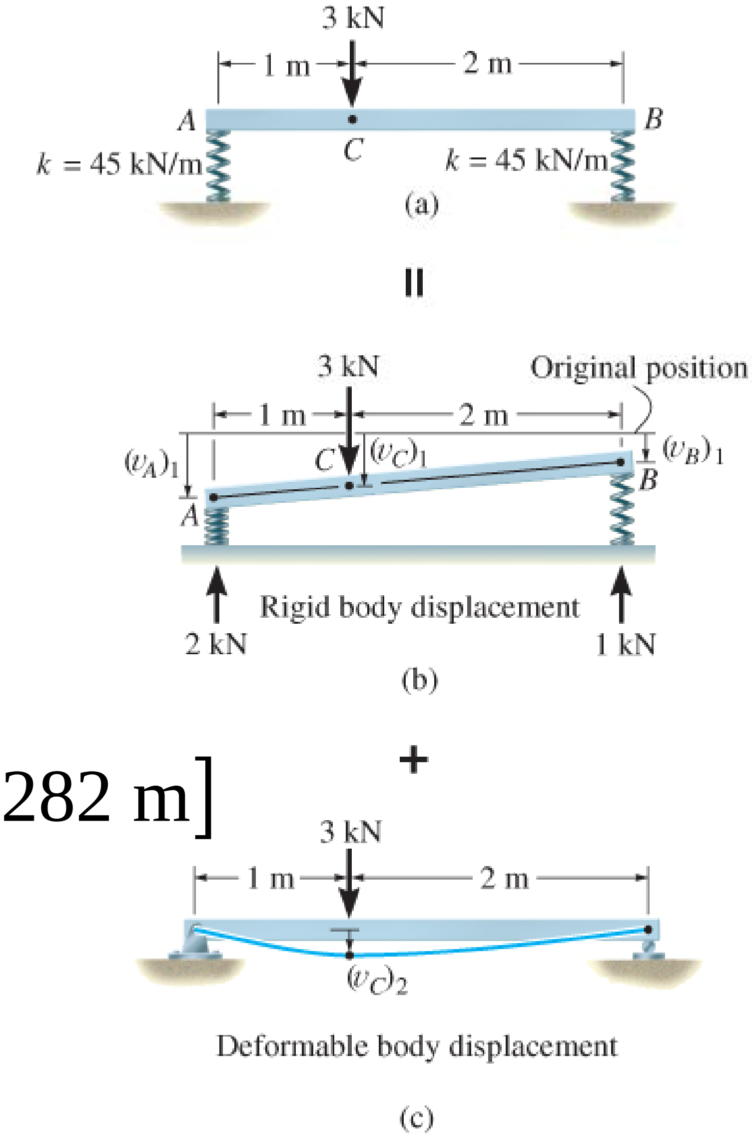
EXAMPLE 12.16 (SOLN)

If bar is considered rigid, these displacements cause it to move into positions shown. For this case, the vertical displacement at C is

$$(\upsilon_C)_1 = (\upsilon_B)_1 + \frac{2\text{ m}}{3\text{ m}}\left[(\upsilon_A)_1 - (\upsilon_B)_1\right]$$

$$= 0.0222\text{ m} + \frac{2}{3}\left[0.0444\text{ m} - 0.0282\text{ m}\right]$$

$$= 0.0370\text{ m} \downarrow$$



EXAMPLE 12.16 (SOLN)

We can find the displacement at C caused by the deformation of the bar, by using the table in Appendix C. We have

$$(\upsilon_C)_2 = \frac{Pab}{6EIL}\left(L^2 - b^2 - a^2\right)$$

$$= \frac{(3\text{ kN})(1\text{ m})(2\text{ m})\left[(3\text{ m})^2 - (2\text{ m})^2 - (1\text{ m})^2\right]}{6(200)\left(10^6\right)\text{kN/m}^2(4.6875)\left(10^{-6}\right)\text{m}^4(3\text{ m})}$$

$$= 1.422\text{ mm}\downarrow$$

EXAMPLE 12.16 (SOLN)

Adding the two displacement components, we get

$$(+\downarrow) \quad \upsilon_C = 0.0370\ \text{m} + 0.001422\ \text{m}$$

$$= 0.0384\ \text{m} = 38.4\ \text{mm}$$