Time Response

- The mathematical representation of a system (*Transfer function or State space*) is used to analyze its transient and steady-state responses to see if these characteristics yield the desired behavior.

- This chapter is devoted to the analysis of ***system transient response***.

Time Response

Poles, Zeros, and System Response

- The output response of a system is the sum of two responses:
  1. *the forced response* (steady-state response or particular solution),
  2. *the natural response* (the homogeneous solution).

Output response = forced response (e.g. constant) + natural response (e.g. exponential)

- **Poles** of a Transfer Function (TF): The values of ***S*** that cause $TF \rightarrow \infty$

roots of the denominator (characteristic polynomial) of the transfer function

- **Zeros** of a TF: the values of ***S*** that cause TF = 0.

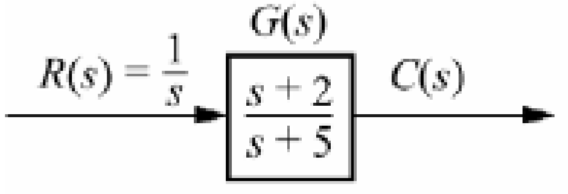
roots of the numerator of the transfer function

Example: Poles and Zeros of a First-Order System

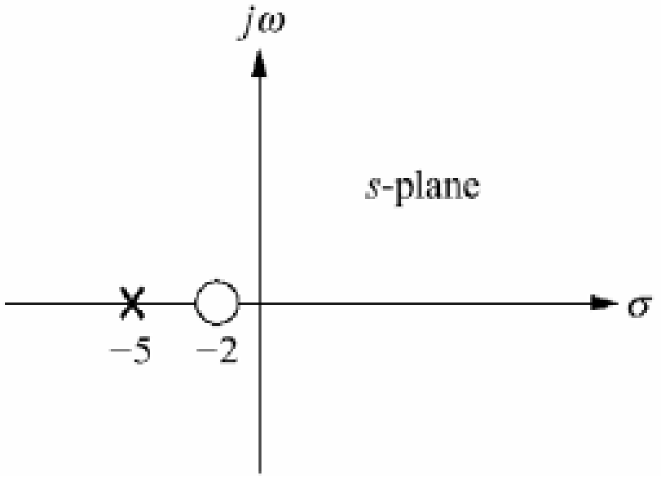
$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

System Output (unit step response)

where $A = \left.\frac{(s+2)}{(s+5)}\right|_{s\to 0} = \frac{2}{5}$ and $B = \left.\frac{(s+2)}{s}\right|_{s\to -5} = \frac{3}{5}$



System showing input and output



pole-zero plot of the system

Inverse Laplace transform: $$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

System Output in time domain (time response)

Example: Poles and Zeros of First Order System

Input function

Input poles $\xrightarrow{\text{generates}}$ *Forced response*

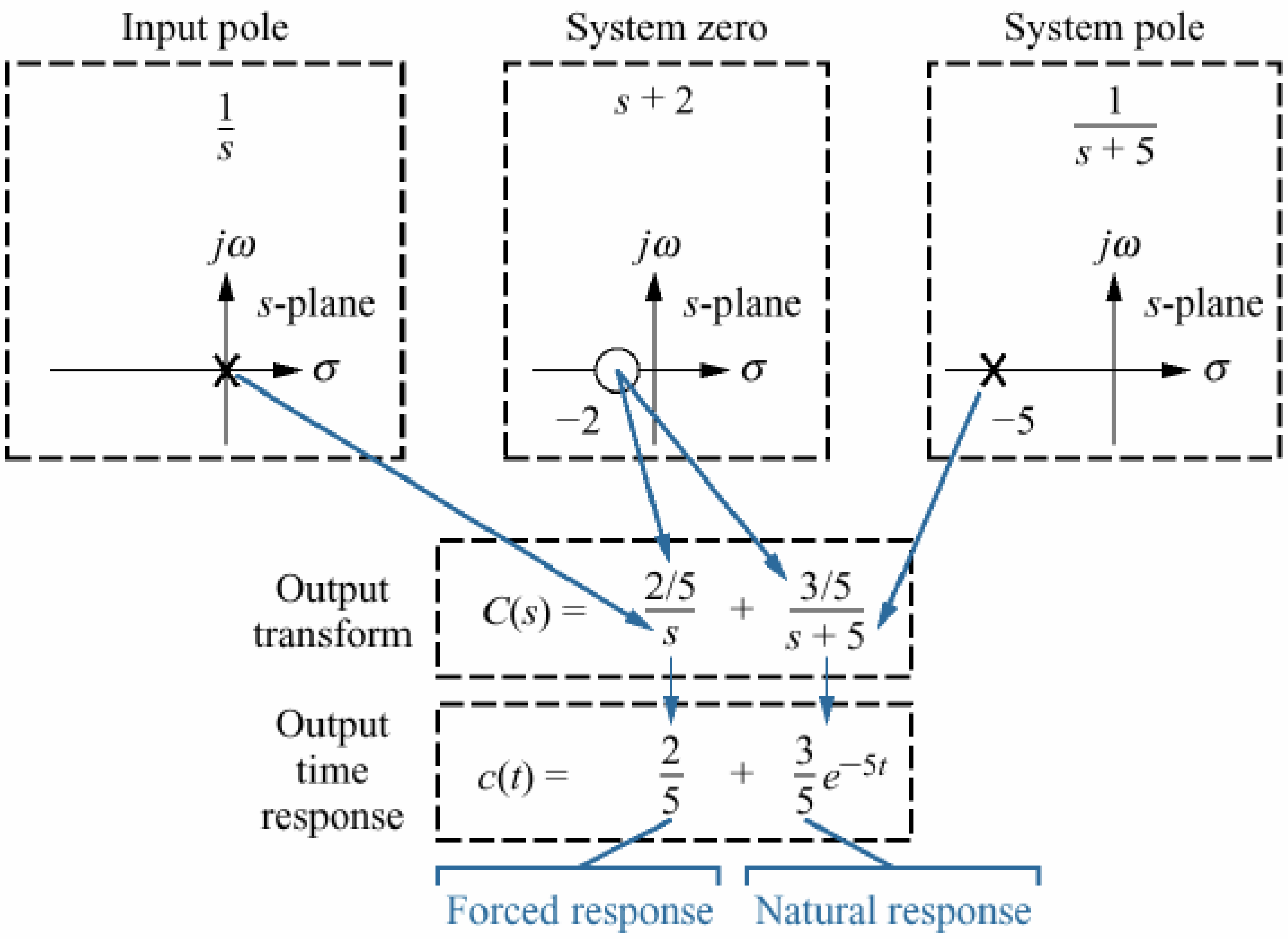
(pole at the origin generated a step function at the output)

System poles $\xrightarrow{\text{generates}}$ *Natural response*

Transfer function

(A pole α on the real axis generates an exponential response of the form $e^{-\alpha t}$ that will decay to zero).

System poles And zeros $\xrightarrow{\text{generates}}$ *Amplitudes* for both the forced and natural responses



Evolution of a system response.

Evaluating Response using Poles

Problem:

Given the following system, write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.

$$R(s) = \frac{1}{s} \longrightarrow \boxed{\frac{(s+3)}{(s+2)(s+4)(s+5)}} \longrightarrow C(s)$$

Solution:

$$C(s) = \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+5)}}_{\text{Natural response}}$$

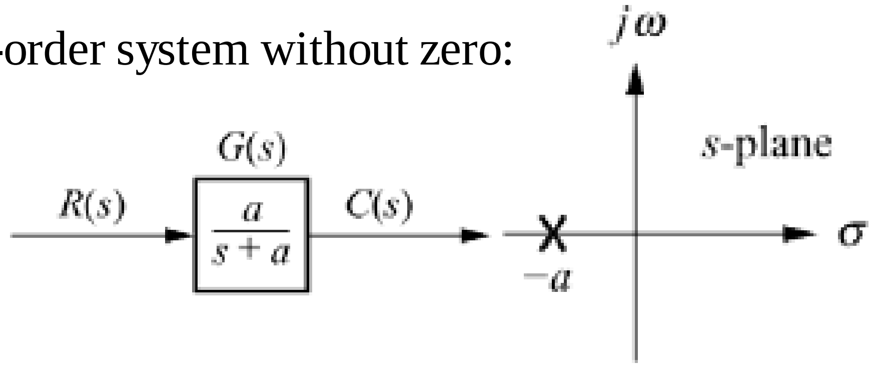
Taking inverse Laplace transform,

$$c(t) = \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}}$$

- Each system pole generates an exponential as part of the natural response.
- The input's pole generates the forced response.

First-Order System: Time Constant

- A first-order system without zero:



- If the input is a unit step: $R(s) = \frac{1}{s}$ then the Laplace transform of the step response is :

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

the input pole at the origin generated the forced response

- Taking the inverse transform

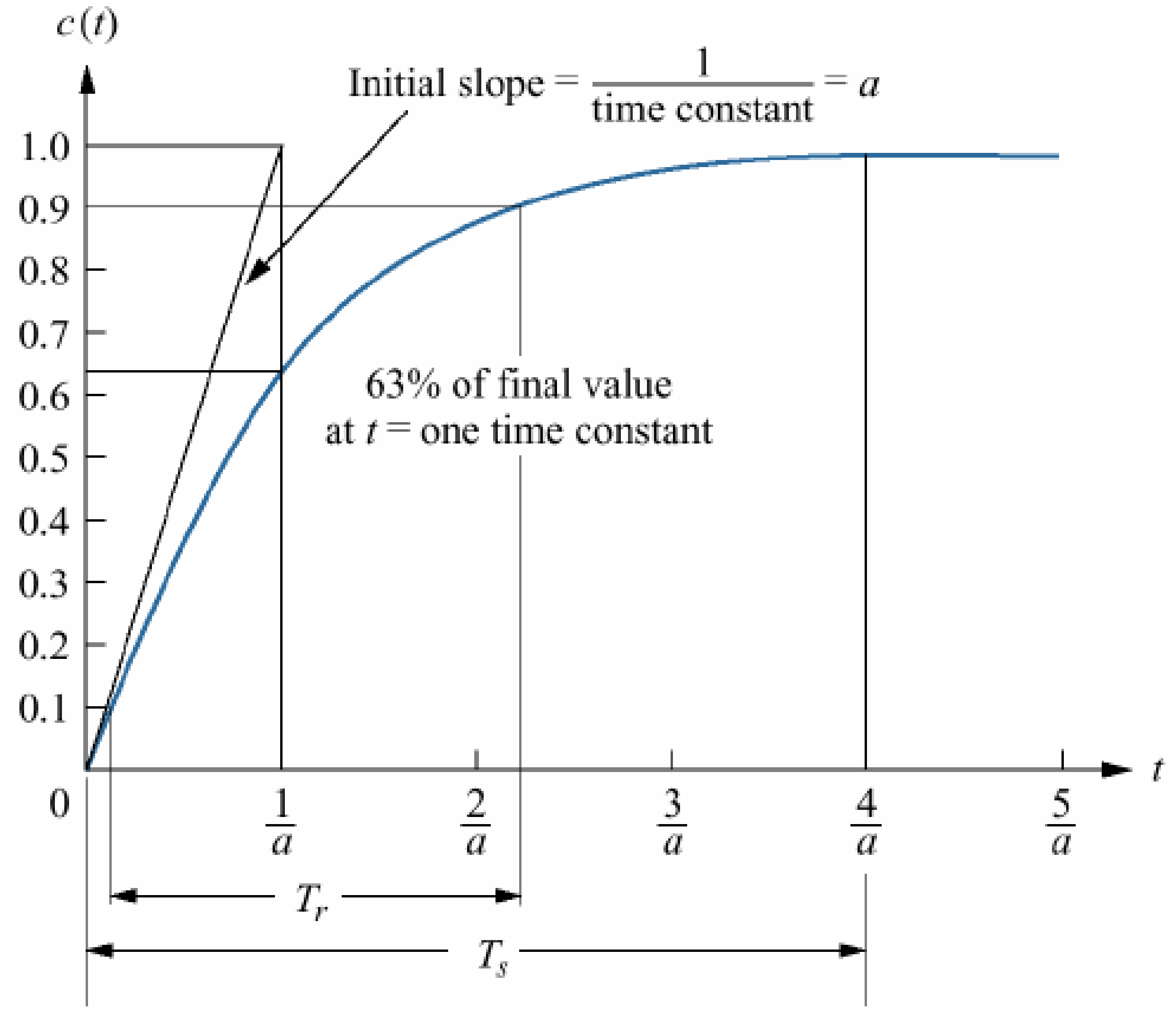
$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

The system pole at –a generated the natural response

- Significance of parameter a (only parameter needed to describe the transient response),

When $\quad t = 1/a, \quad e^{-at}\Big|_{t=1/a} = e^{-1} = 0.37$

Hence, $\quad c(t)\Big|_{t=1/a} = 1 - 0.37 = 0.63$



Some Terminology

(three transient response performance specifications).

1. ***Time constant T_c:*** Time it takes for the step response to rise to 63% of its final value.

$$\boxed{T_c = \frac{1}{a}}$$

- we can call the parameter *a* (*system pole*) the *exponential frequency* (The reciprocal of the time constant)
- T_c is related to the speed at which the system responds to a step input.

2. ***Rise Time T_r:*** Rise time is defined as the time for the response to go from *0.1* to *0.9* of its final value.

found by solving for the difference in time at *c(t)* = 0.9 and *c(t)* = 0.1

$$C(t_2) = 0.9 = 1 - e^{-at_2} \Rightarrow t_2 = -\frac{\ln(0.1)}{a} = \frac{2.31}{a}$$

$$C(t_1) = 0.1 = 1 - e^{-at_1} \Rightarrow t_1 = -\frac{\ln(0.9)}{a} = \frac{0.11}{a}$$

$$\Rightarrow T_r = t_2 - t_1 = \frac{2.13}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

time for $C(t) = 0.1$

time for $C(t) = 0.9$

Rise time: $\boxed{T_r = \frac{2.2}{a}}$

3. ***Settling time T_s:*** The time for the response to reach, and stay within, 2% of its final value.

Letting $C(t) = 0.98$ and solving for time, t, we find the settling time to be

$$C(T_s) = 0.98 = 1 - e^{-at_2} \Rightarrow T_s = -\frac{\ln(0.98)}{a} = \frac{4}{a}$$

$$\boxed{T_s = \frac{4}{a}}$$

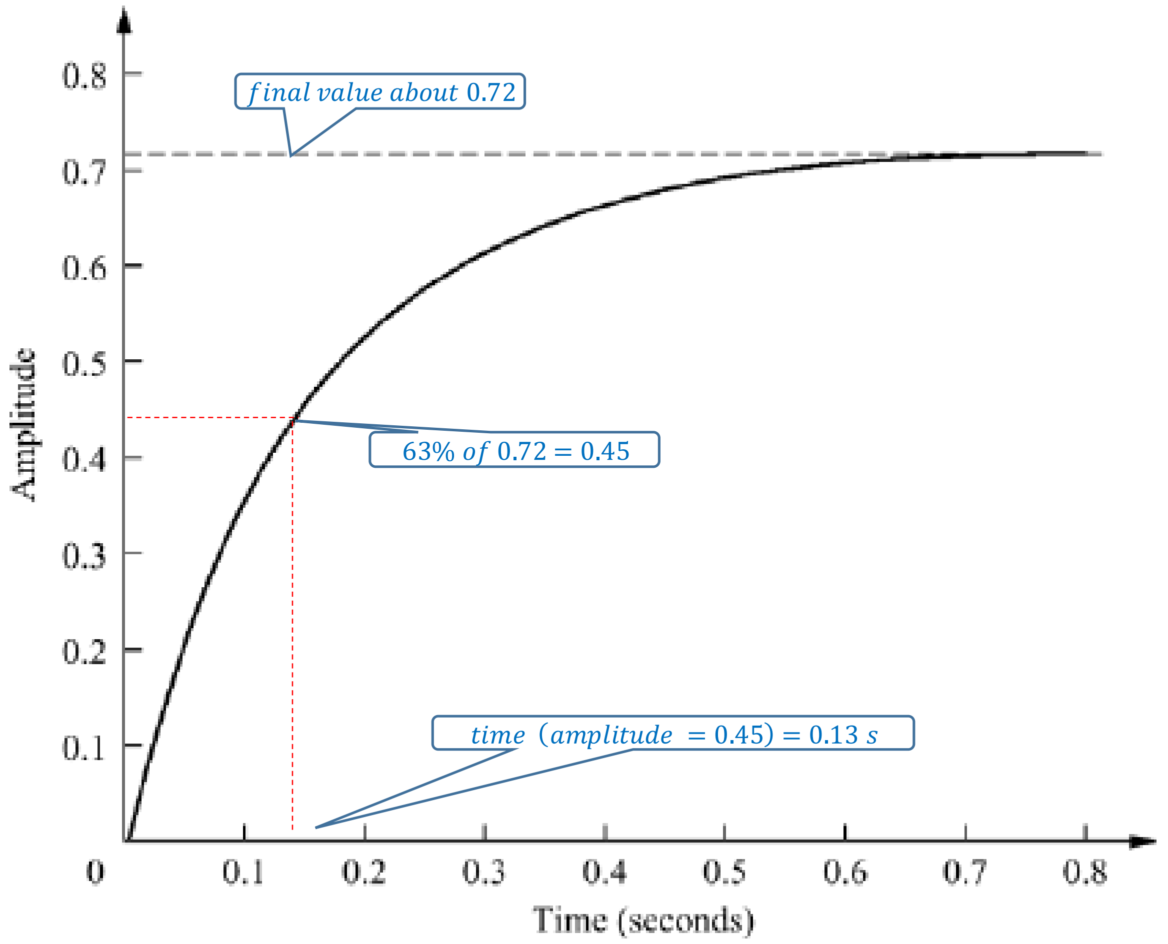
First-Order Transfer Function via Testing

- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

- A simple first order system has : $G(s) = K / (s+a)$,

  and step response is: $C(s) = \dfrac{K}{s(s+a)} = \dfrac{K/a}{s} - \dfrac{K/a}{(s+a)}$

- From *the response*, we identify K and a to obtain the transfer function.



To find a:

63% — final value about 0.72

$$Time\ constant = Time(0.63 \times 0.72)$$

$$= Time(0.45) = 0.13\ second$$

$$a = {}^{1}/_{0.13} = 7.7$$

To find K: The forced response reaches a steady-state value of

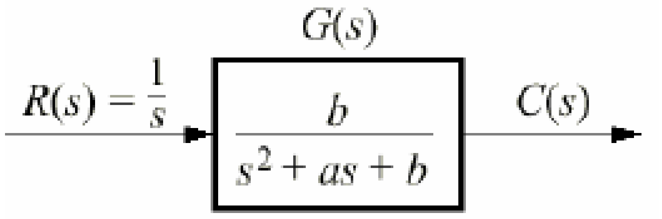
$^{K}/_{a} = 0.72$ ⇒ $K = 5.54$

Transfer function, $\boxed{G(s) = \dfrac{5.54}{(s+7.7)}}$

Second-Order System

- Parameters of First-order system determine the *speed* of the system.
- Parameters of Second-order system determine the *form (shape)* of the system.
- Some examples of second-order response:

  By assigning appropriate values to parameters a and b, we can show all possible second-order transient responses.



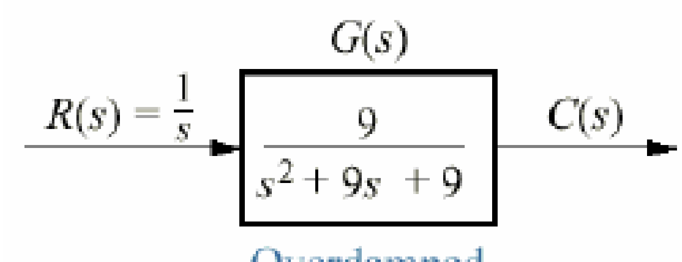
the general case

(two finite poles and no zeros)

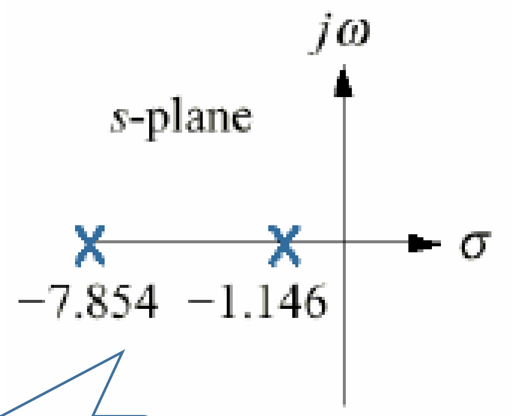
1. Over-damped Response: (pole at the origin from the unit step input and Two real poles that come from the system)

$$C(s) = \frac{9}{s(s^2+9s+9)} = \frac{9}{s(s+7.854)(s+1.146)}$$

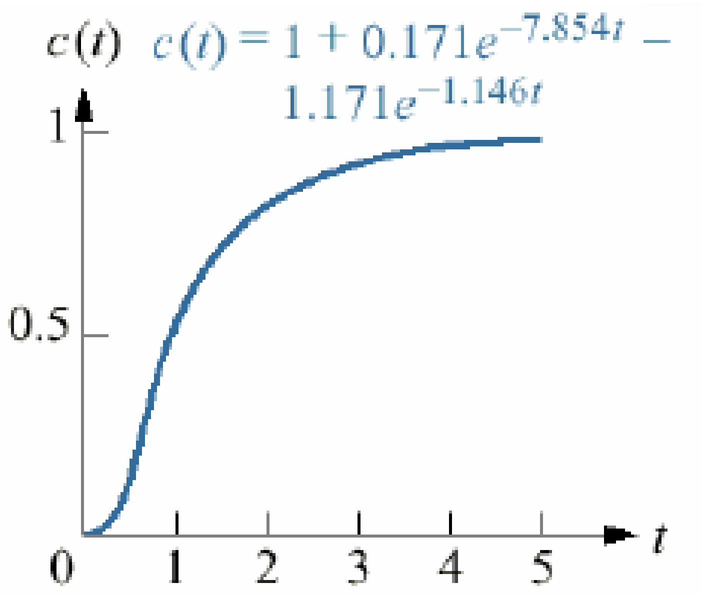
$a = 9$ *and* $b = 9$



Overdamped



Two real poles



Second-Order System

2. Under-damped Response: (pole at the origin from the input and Two complex poles that come from the system).

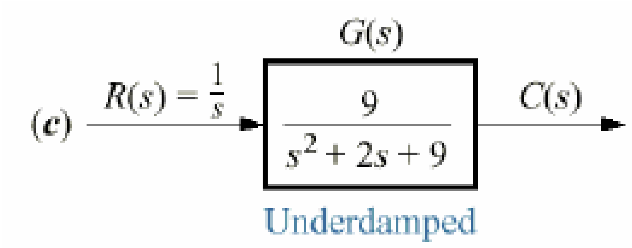
$$C(s)=\frac{9}{s(s^2+2s+9)}=\frac{9}{s(s+[1+j\sqrt{8}])(s+[1-j\sqrt{8}])}$$

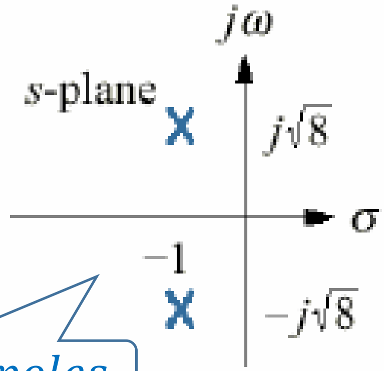
$a = 2$ and $b = 9$

Poles from the system: $s=-1\pm j\sqrt{8}$

exponential decay

frequency of the sinusoidal oscillation.

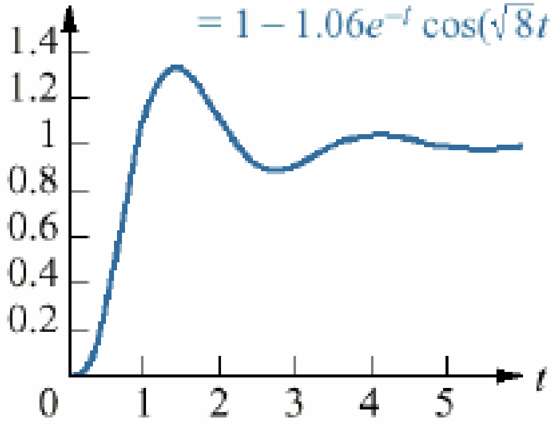




Two complex poles

$c(t)=1-e^{-t}(\cos\sqrt{8}t+\frac{\sqrt{8}}{8}\sin\sqrt{8}t)$

$=1-1.06e^{-t}\cos(\sqrt{8}t-19.47^\circ)$

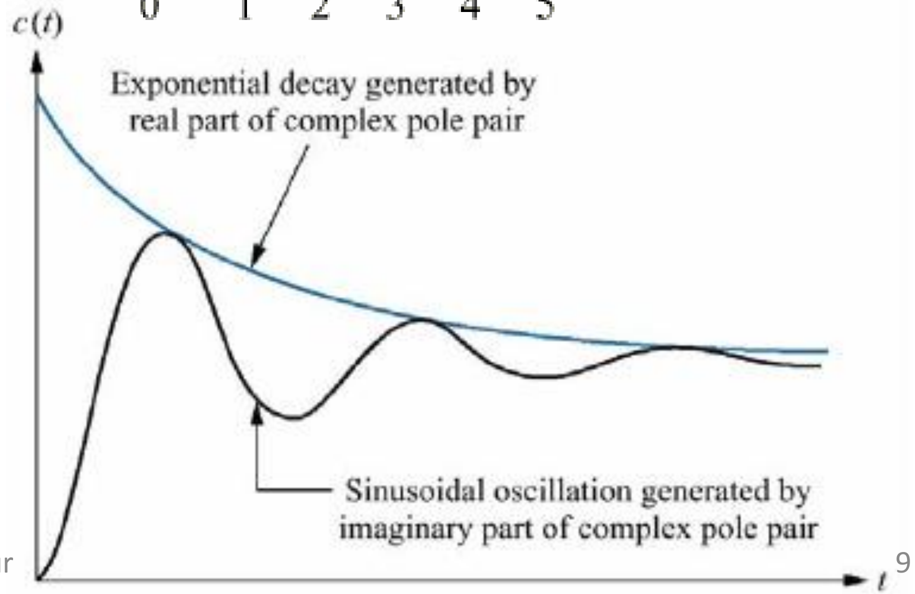


$$c(t)=K_1+e^{-t}(K_2\cos\sqrt{8}t+K_3\sin\sqrt{8}t)$$
$$=K_1+K_4e^{-t}(\cos\sqrt{8}t-\varphi)$$

Where,

$$\varphi=\tan^{-1}\frac{K_3}{K_2},\ \text{and}\ K_4=\sqrt{K_2^{\,2}+K_3^{\,2}}$$

Second-order step response components generated by complex poles



Second-Order System

3. Un-damped Response: pole at the origin that comes from the input and two imaginary poles that come from the system.

$$C(s) = \frac{9}{s(s^2+9)}$$

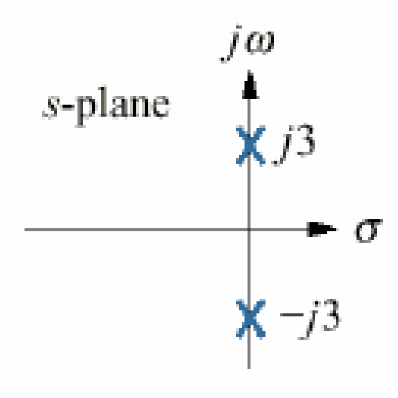
$a = 0$ and $b = 9$

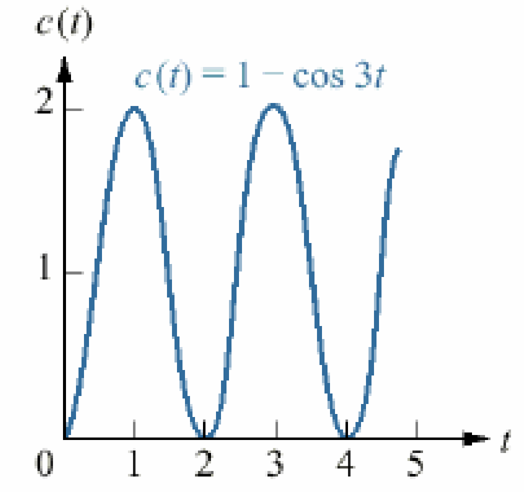
$$s = \pm j3$$

two system poles on the imaginary axis

$$c(t) = K_1 + K_4 \cos(3t - \varphi)$$

There is no exponential term, so no decay.





4. Critically Damped Response: pole at the origin that comes from the input and two multiple real poles that come from the system.

$$C(s) = \frac{9}{s(s^2+6s+9)} = \frac{9}{s(s+3)^2}$$

$a = 6$ and $b = 9$

$$s = -3, -3$$

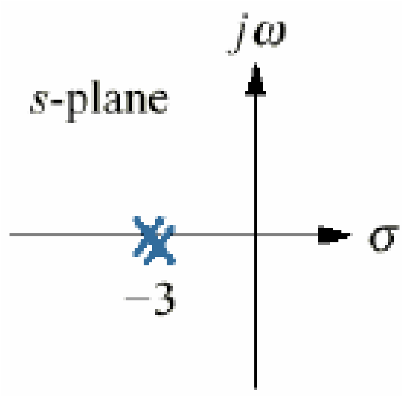
two poles (double) on the real axis at −3

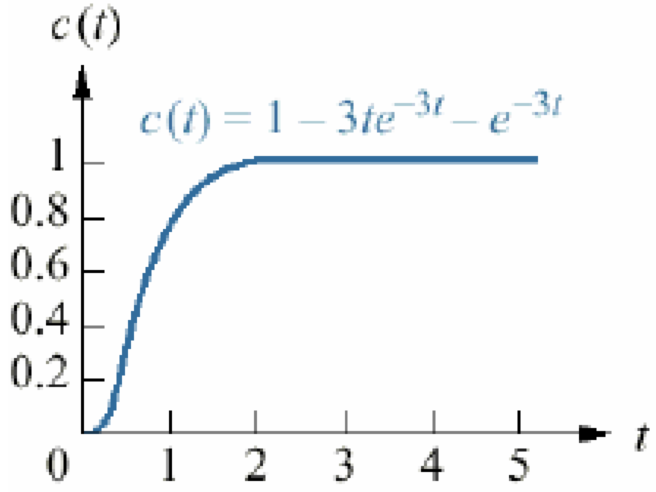
input pole

system poles

$$c(t) = K_1 + K_2 t e^{-3t} + K_3 t e^{-3t}$$

There is no sinusoidal term, so no oscillation.





Second-Order System

All Together

Over-damped responses

Two real poles at $-\sigma_1, -\sigma_2$

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

Under-damped responses

Two complex poles at $-\sigma_d \mp jw_d$

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$

Un-damped responses

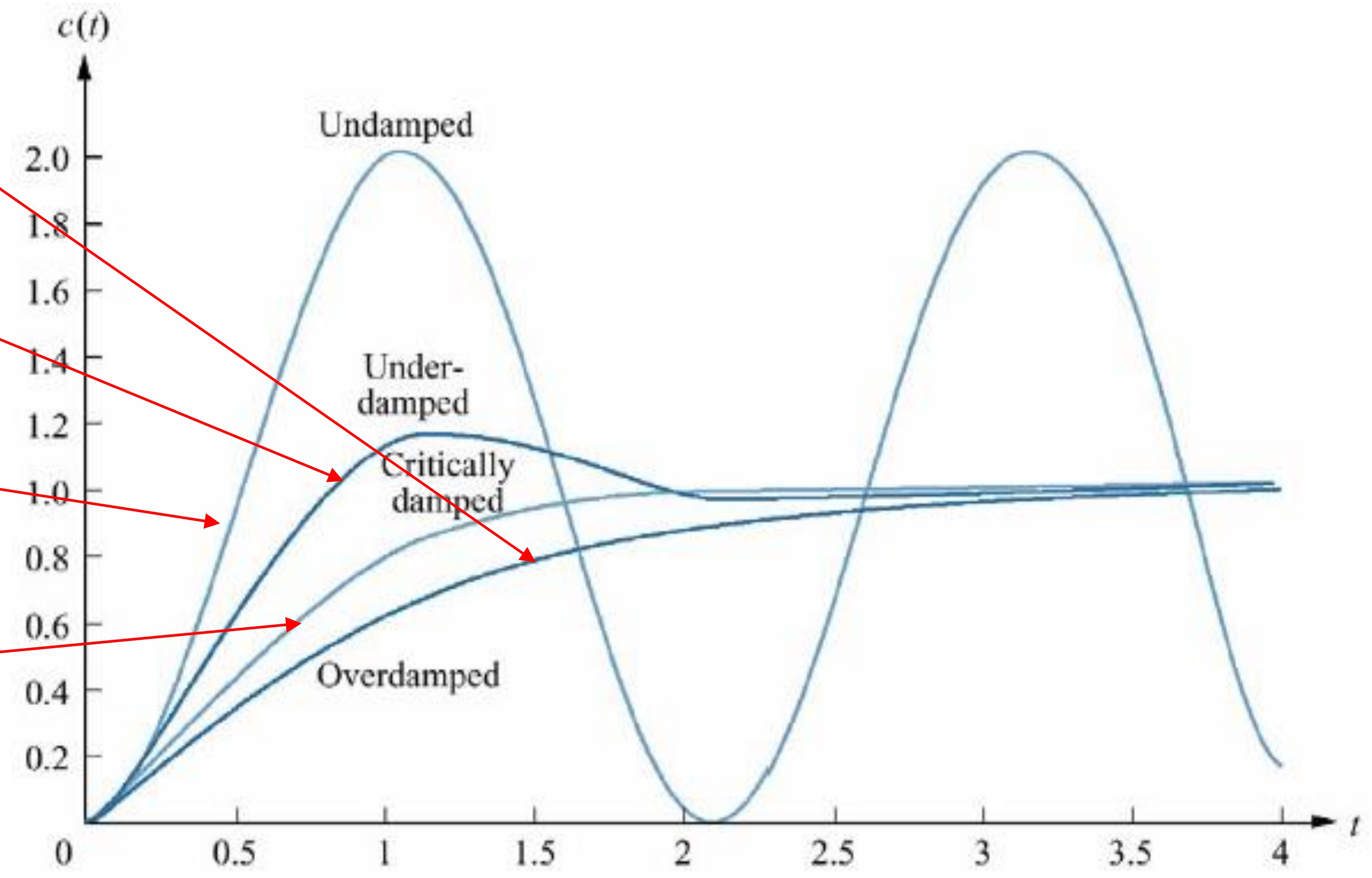
Two imaginary poles at $\mp jw_1$

$$c(t) = A\cos(\omega_1 t - \phi)$$

Critically damped responses

Two real poles at $-\sigma_1$

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$



Step responses for second-order system damping cases

The General Second-Order System

- we define two physically meaningful specifications to describe the characteristics of the second-order transient response systems, *natural frequency* and *damping ratio*.

Natural Frequency ω_n : the frequency of oscillation of the system without damping.

Damping Ratio ξ : dimensionless measure describing how oscillations in a system decay.

$$\xi = \frac{\text{Exponential Decay Frequency}}{\text{Natural Frequency (rad/second)}} = \frac{1}{2\pi}\frac{\text{Natural period (seconds)}}{\text{Exponential time constant}}$$

- Generalization of second-order system in terms of ξ and ω_n.

Consider the general system, $G(s) = \dfrac{b}{s^2 + as + b}$

For un-damped(without damping) system, $a = 0$, and the poles are on $j\omega - axis$ at $\mp j\sqrt{b}$, ← $G(s) = \dfrac{b}{s^2 + b}$

$\therefore \omega_n = \sqrt{b}$, Hence $b = \omega_n{}^2$

For an under-damped system, poles have real part $\sigma = -{}^{a}/_{2}$ (exponential decay)

$$\xi = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n} \Rightarrow a = 2\xi\omega_n$$

Exponential decay frequency

Natural frequency

The general second-order transfer function finally looks like

$$\boxed{G(s) = \frac{\omega_n{}^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}}$$

Second-Order System As a Function of Damping Ratio

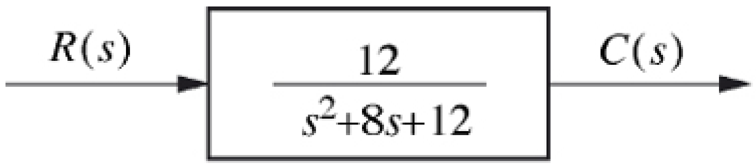
- Relationship between the quantities $\boldsymbol{\omega_n}$ ***and*** $\boldsymbol{\xi}$ and the pole location.

Solving for the poles of the transfer function

$$G(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \xrightarrow{\text{Denominator roots}} s_{1,2}=-\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

Example

For the system find the value of $\boldsymbol{\xi}$ and report the kind of response expected.

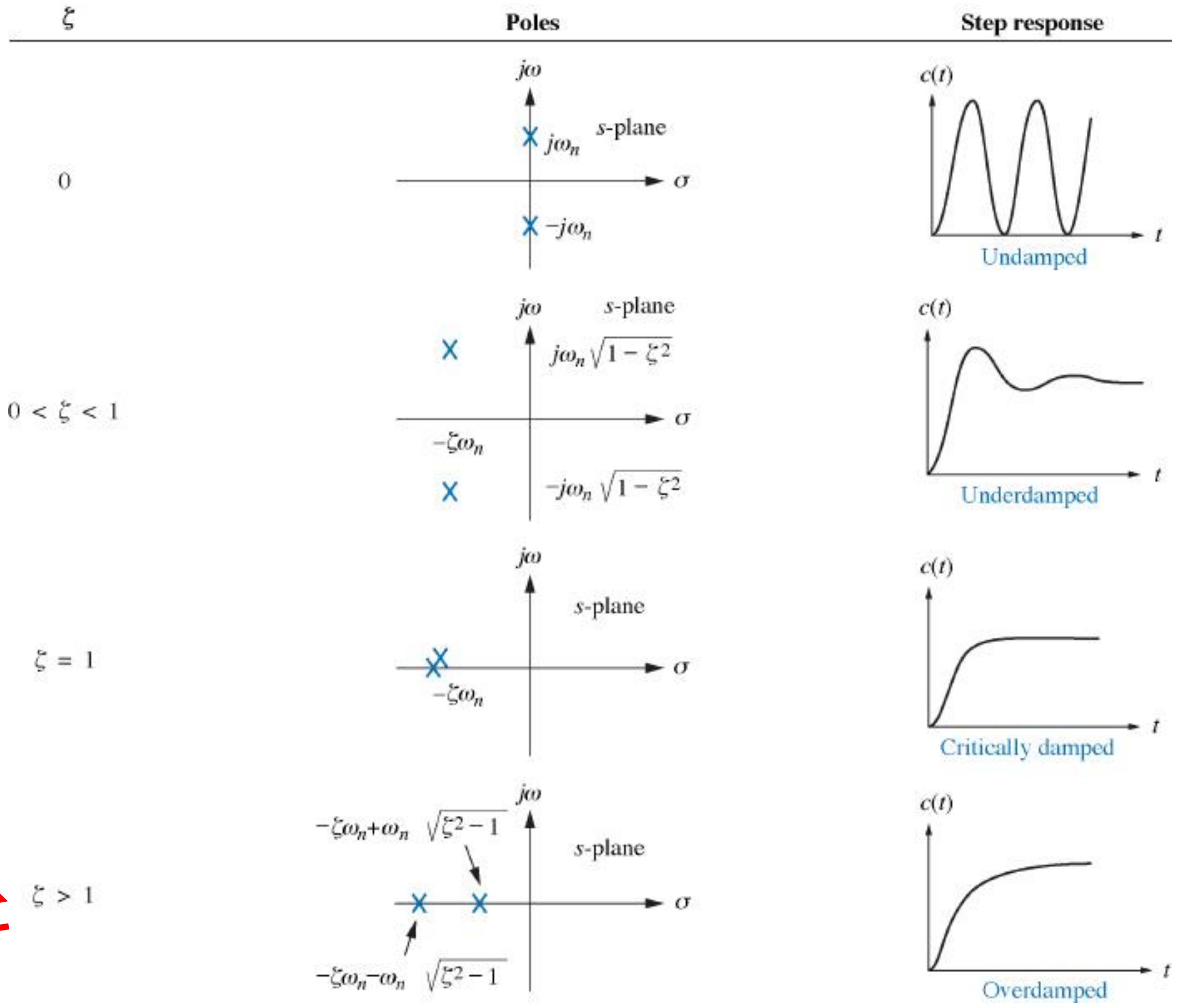


We have $G(s)=\dfrac{\omega_n^2}{s^2+2\boldsymbol{\xi}\omega_n s+\omega_n^2}=\dfrac{12}{s^2+8s+12}$

$\Rightarrow \quad \omega_n^2=12 \quad \Rightarrow \quad \omega_n=\sqrt{12}$

and $\quad 2\boldsymbol{\xi}\omega_n=8 \quad \Rightarrow \quad \boldsymbol{\xi}=\dfrac{8}{2\sqrt{12}}=\dfrac{2}{\sqrt{3}}>1$

System is over-damped.



Second-order response as a function of damping ratio

Underdamped Second-Order Systems

- The nature of the response obtained is related to the value of the damping ratio **ξ** (over-damped, critically damped, underdamped, and un-damped responses.).

- Step response for the general second-order system,

$$C(s) = R(s)G(s) = \frac{1}{s}\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \Rightarrow C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Expanding by partial fractions, ($\xi < 1$ the underdamped case)

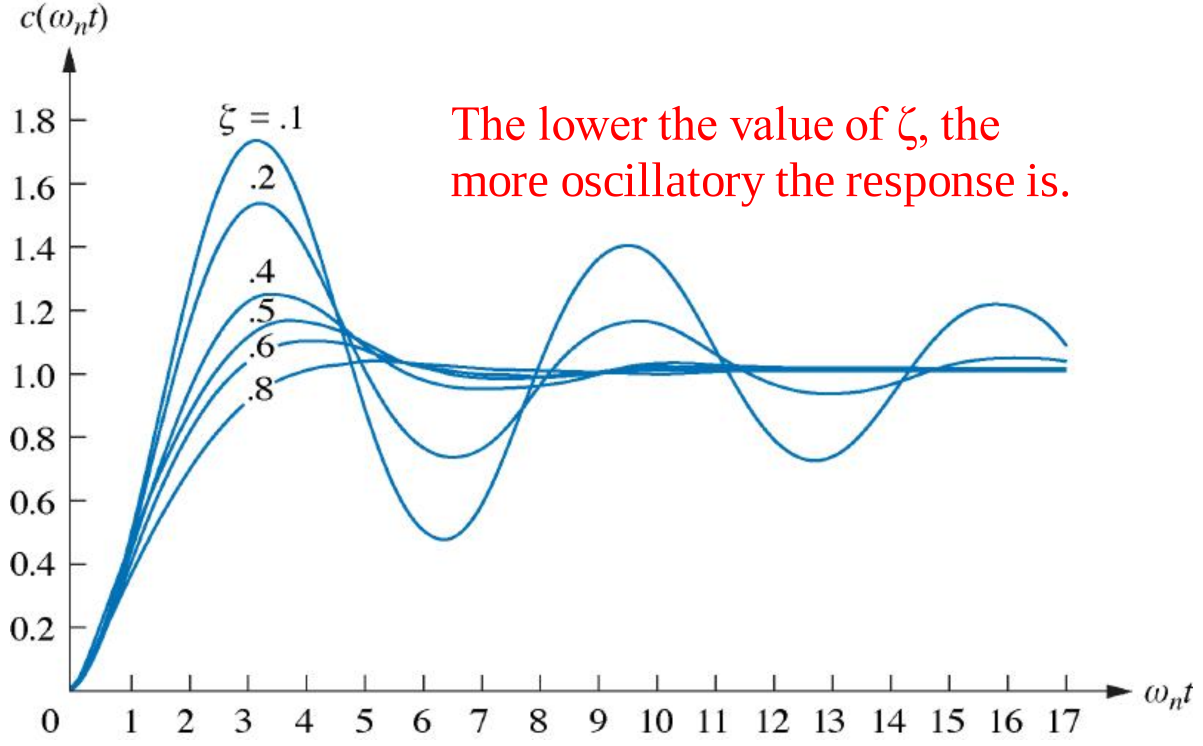
$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

inverse Laplace transform

$$c(t) = 1 - e^{-\xi\omega_n t}\left(\cos\omega_n\sqrt{1-\xi^2}\,t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_n\sqrt{1-\xi^2}\,t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\cos\left(\omega_n\sqrt{1-\xi^2}\,t - \varphi\right)$$

Where, $\varphi = \tan^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)$



The lower the value of ζ, the more oppscillatory the response is.

Second-order underdamped responses for damping ratio values

Underdamped Second-Order Systems

Specifications

- Other parameters associated with the underdamped response are *rise time*, *peak time*, *percent overshoot*, and *settling time*.

Rise time T_r: The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.

Peak time T_p: The time required to reach the first, or maximum, peak. $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$

percent overshoot $\%OS$: The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.

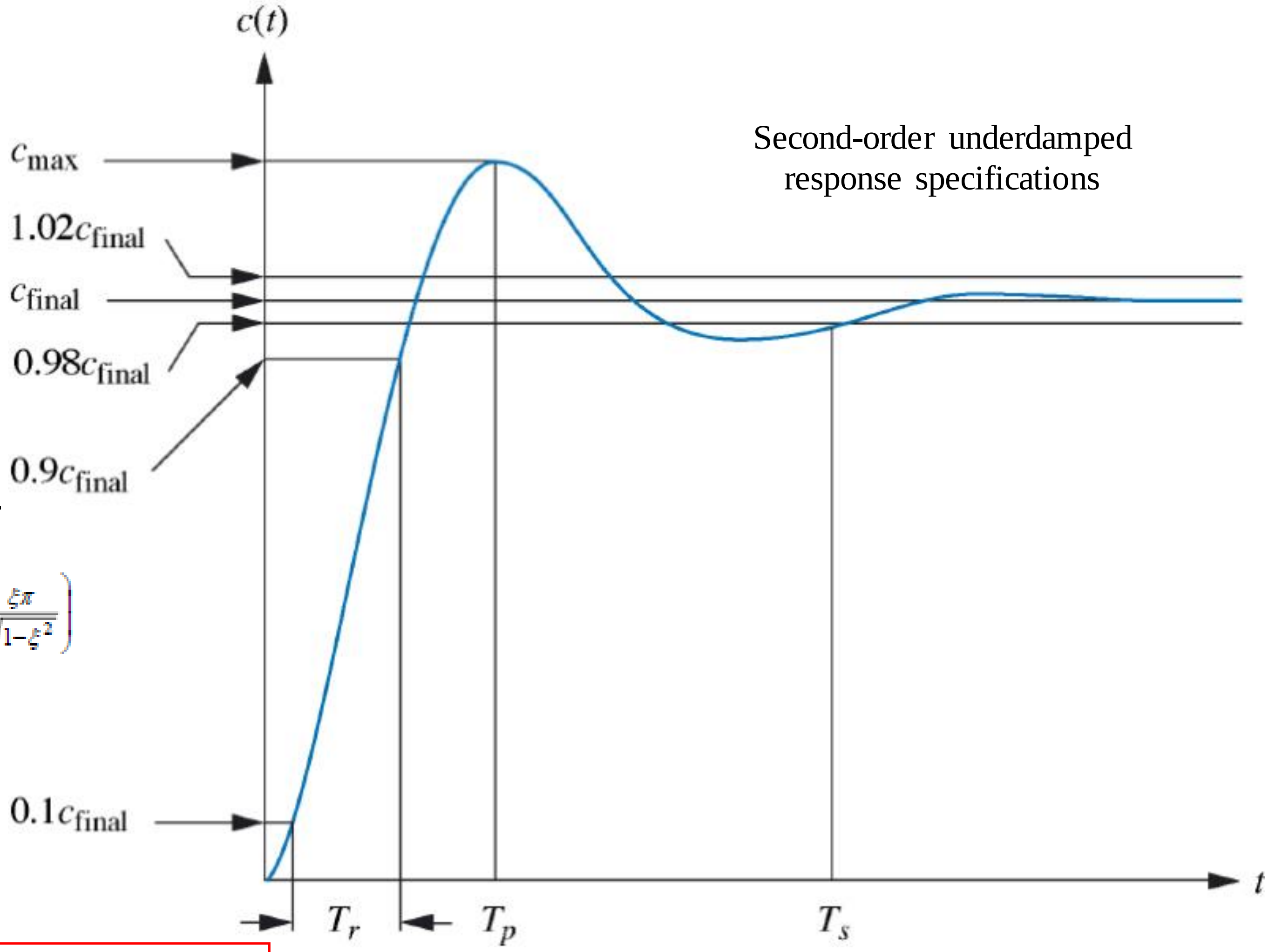
$$\%OS = \frac{c_{\max} - c_{final}}{c_{final}} \times 100$$

$c_{\max} = c(T_p) = 1 + e^{-\left(\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)}$ and $c_{final} = 1$

settling time T_s: The time required for the transient's damped oscillations to reach and stay within ±2% of the steady-state value.

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\xi^2}\right)}{\xi\omega_n} \approx \frac{4}{\xi\omega_n}$$

Derivation: ***self study***.



Under-damped Second-Order Systems

Specifications (continued)

from the Pythagorean theorem

$$\omega_d^2 + \sigma_d^2 = \omega_n^2$$
$$\sigma_d = \xi\omega_n$$

$$\Rightarrow \quad \omega_d = \sqrt{\omega_n^2 - \sigma_d^2} = \sqrt{\omega_n^2 - \xi^2\omega_n^2} = \omega_n\sqrt{1 - \xi^2}$$

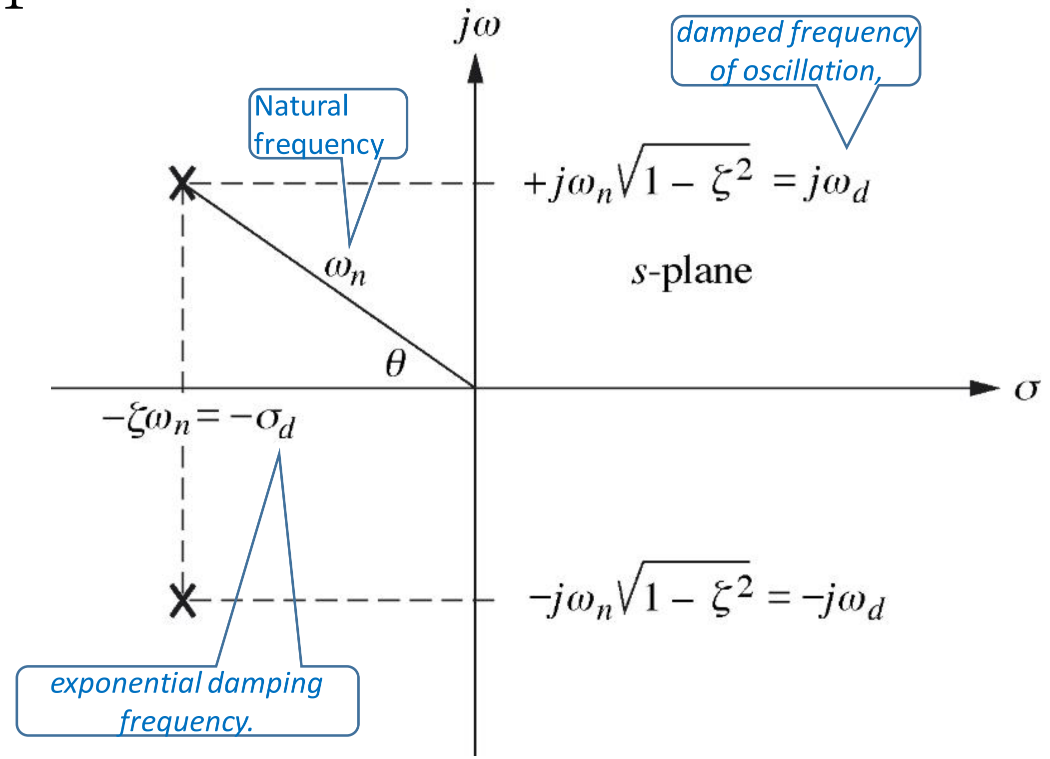
$$\cos(\theta) = \frac{\sigma_d}{\omega_n} = \frac{\xi\omega_n}{\omega_n} = \xi$$

inversely proportional to the imaginary part of the pole.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \quad \Rightarrow \quad T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_d}$$

inversely proportional to the real part of the pole.

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\xi^2}\right)}{\xi\omega_n} \approx \frac{4}{\xi\omega_n} \quad \Rightarrow \quad T_s = \frac{4}{\xi\omega_n} = \frac{4}{\sigma_d}$$

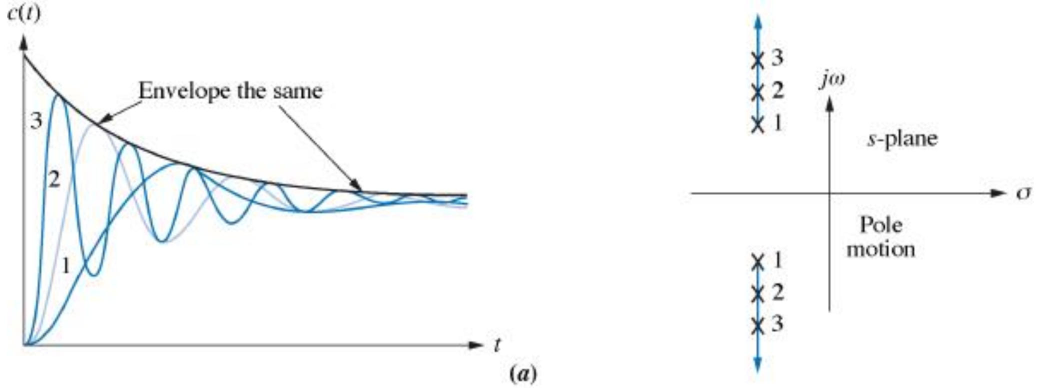


Under-damped Second-Order Systems

Step Response as Pole moves

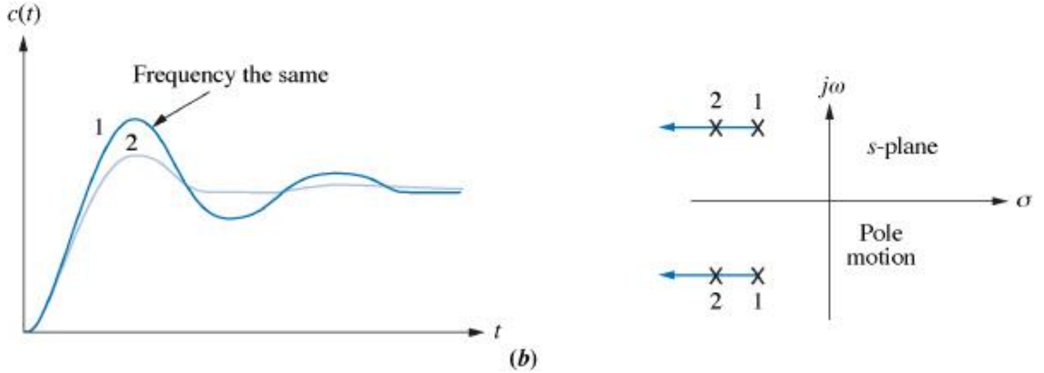
poles move in a vertical direction (with constant real part)

- frequency increases
- envelope remains the same (constant real part)
- settling time is virtually the same
- overshoot increases, the rise time decreases



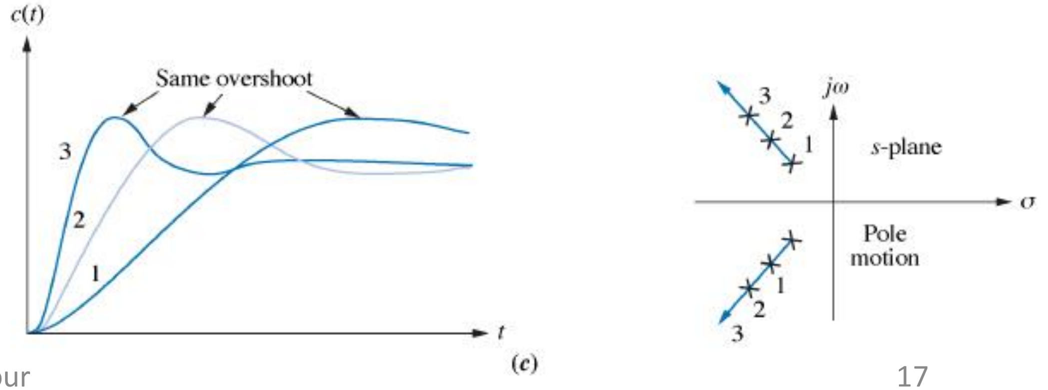
poles move in a horizontal direction (with constant imaginary part)

- As the poles move to the left, response damps out more rapidly.
- peak time is the same for all waveforms (constant imaginary part)



poles move in along a constant radial line direction

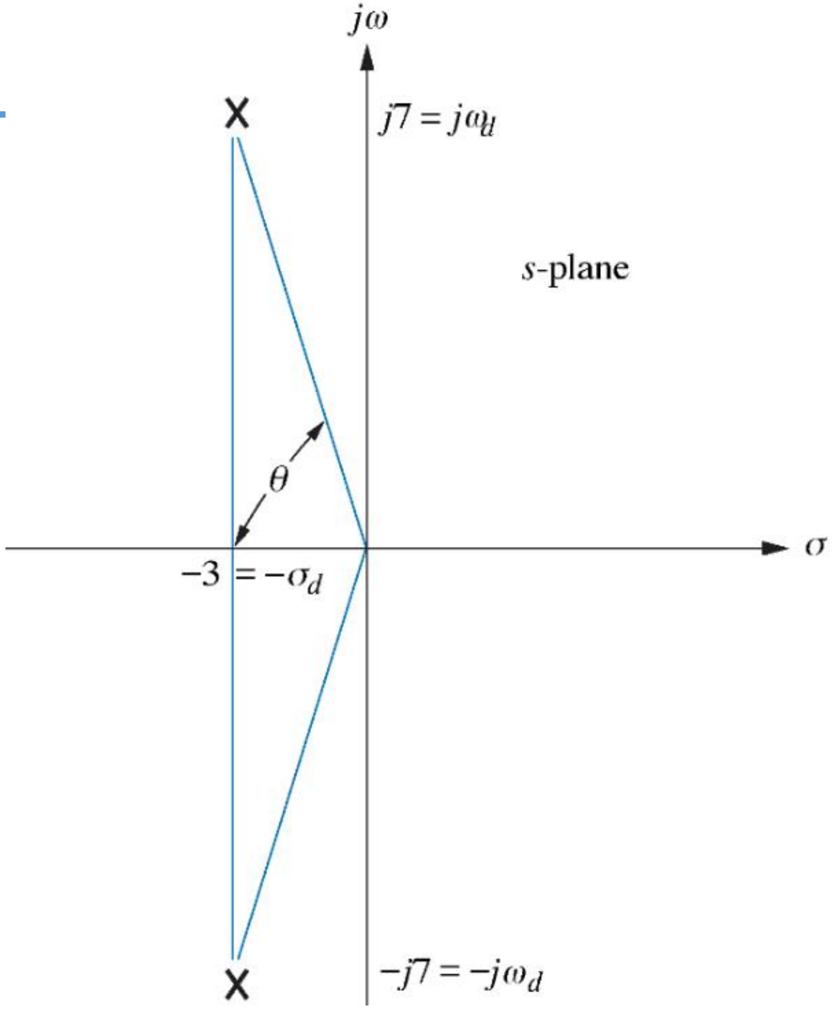
- The percent overshoot remains the same.
- The farther the poles are from the origin, the more rapid the response.



Finding TP, %OS, and TS From Pole Location

Problem: Given the pole plot find **ξ**, ω_n, T_p, *%OS*, and T_s.

Solution:



Damping ration, $\zeta = \cos(\theta) = \cos[\arctan(7/3)] = 0.394.$

Natural frequency, $\omega_n = \sqrt{\omega_d^2 + \sigma_d^2} = \sqrt{7^2 - 3^2} = 7.616$

Peak time, $T_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{7} = 0.449 \text{ second}$

Percent overshoot, $\%OS = e^{-\left(\xi\pi / \sqrt{1-\xi^2}\right)} \times 100 = 26\%$

The approximate settling time, $T_s = \dfrac{4}{\sigma_d} = \dfrac{4}{3} = 1.333 \text{ second}$

System Response with Additional Poles

- If a system has more than two poles or has zeros, we cannot use the formulas to calculate the performance specifications that we derived.

- We need to approximate that system to a second-order system that has just two dominant complex poles.

Assuming two complex poles at $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ And the real pole at $-\alpha_r$

Time domain step response, $c(t) = Au(t) + e^{-\xi\omega_n t}\left(B\cos\omega_d t + C\sin\omega_d t\right) + De^{-\alpha_r t}$

Case I: α_r is not much larger than $\zeta\omega_n$.

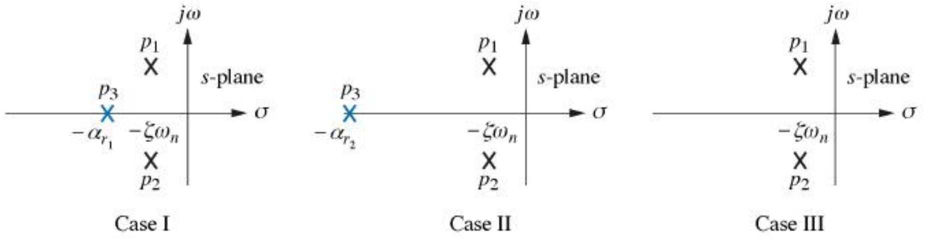
× Can't be approximated as 2nd-order system.

Case II: α_r is much larger than $\zeta\omega_n$.

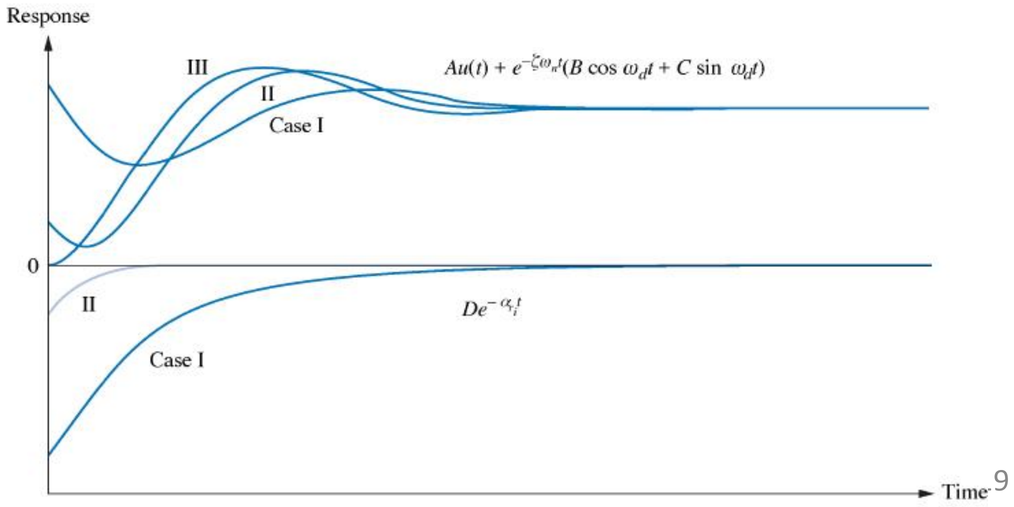
Can be approximated as 2nd-order system.

Case III: $\alpha_r = \infty$.

Pure 2nd-order system.



(a)



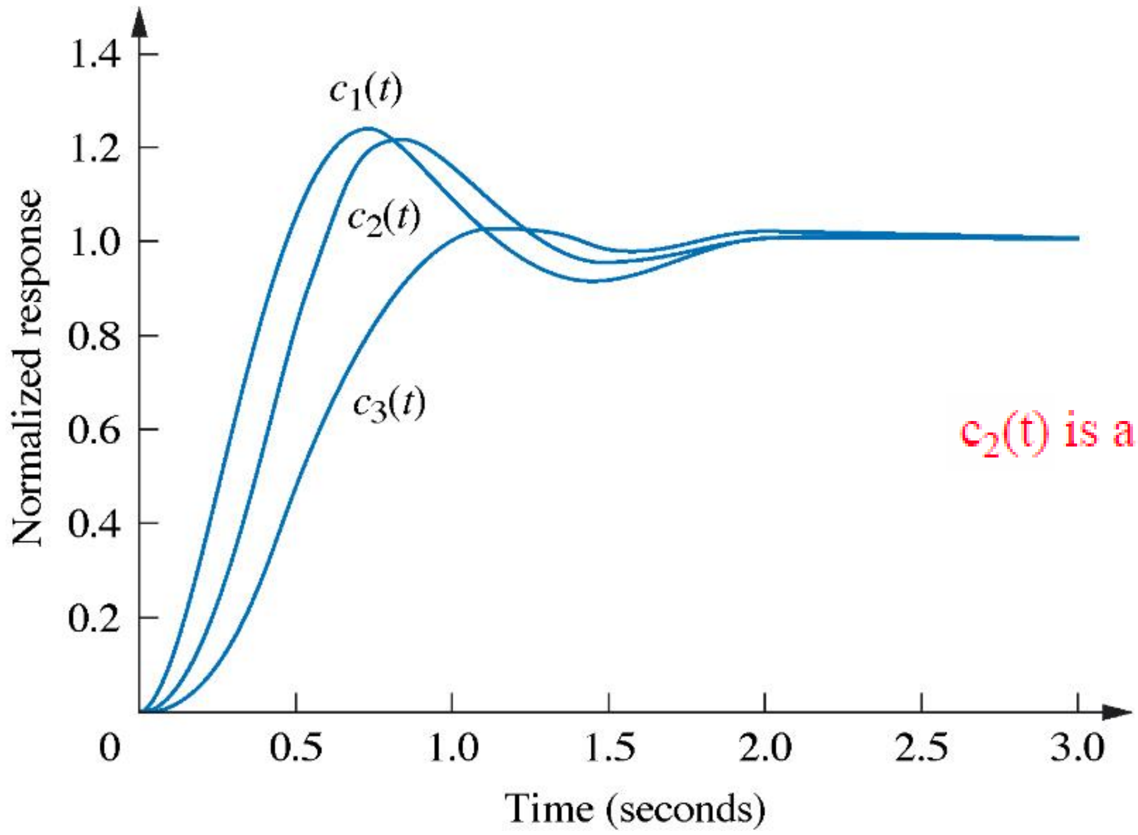
(b)

Comparing Responses of Three-Pole Systems

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542}$$ → 2^{nd}-order system

$$T_2(s) = \frac{24.542}{(s+10)(s^2 + 4s + 24.542)}$$ → 3^{rd}-order system, nondominant pole at -10

$$T_3(s) = \frac{24.542}{(s+3)(s^2 + 4s + 24.542)}$$ → 3^{rd}-order system, nondominant pole at -3



$c_2(t)$ is a better approximation of $c_1(t)$

Evaluating Pole-Zero Cancellation

Problem: For any function for which pole-zero cancellation is valid, find the approximate response.

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)}$$

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)}$$

Zero is at -4

Solution:

The partial-fraction expansion of $C_1(t)$ is

$$C_1(s) = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{1}{s+3.5}$$

Residue = 1

Nearest pole = -3.5 close to zero

That residue (1) is not negligible. So a 2nd-order step response approximation cannot be made for $C_1(t)$.

The partial-fraction expansion of $C_2(t)$ is

$$C_2(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \cancel{\frac{0.033}{s+4.01}}$$

Residue = 0.033

Nearest pole

That residue (0.033) is negligible, so cancel zero and that pole.

Hence, the approximate response, $c_2(t) = 0.87 - 5.3e^{-5t} + 4.4e^{-6t}$