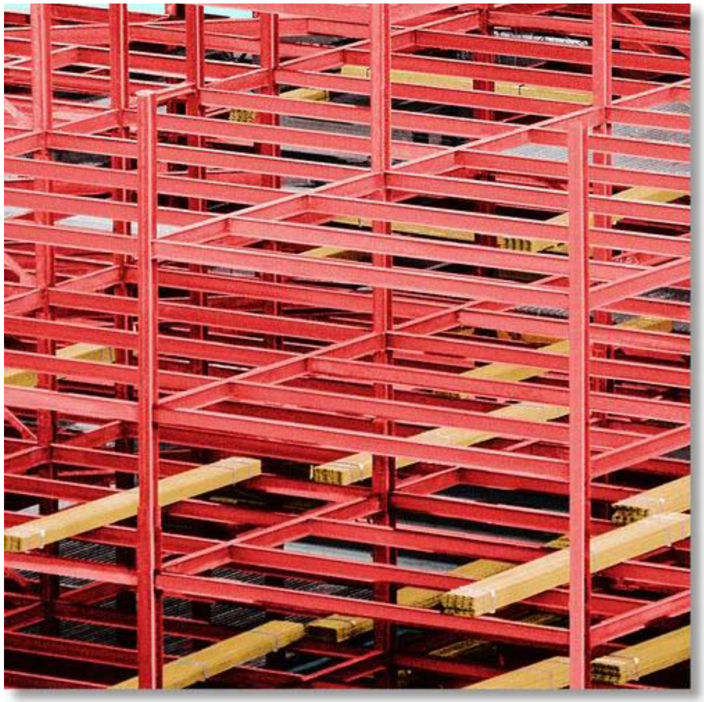
CHAPTER OBJECTIVES



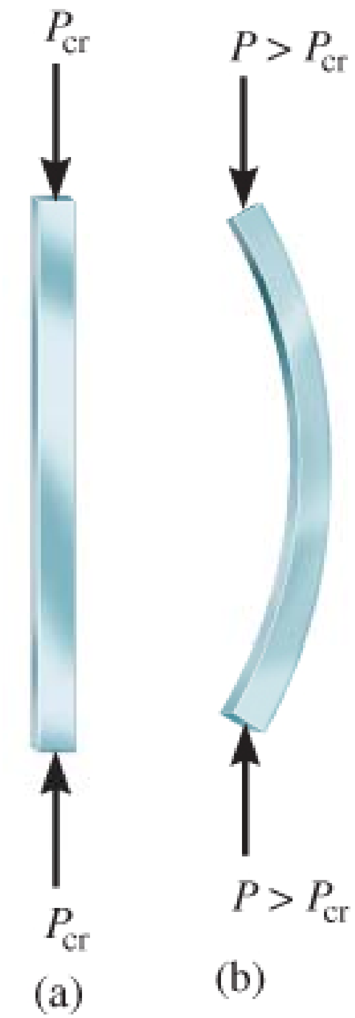
- Discuss the behavior of columns.
- Discuss the buckling of columns.
- Determine the axial load needed to buckle an ideal column.
- Analyze the buckling with bending of a column.
- Discuss methods used to design concentric and eccentric columns.

CHAPTER OUTLINE

1. Critical Load
2. Ideal Column with Pin Supports
3. Columns Having Various Types of Supports
4. *Design of Columns for Concentric Loading
5. *Design of Columns for Eccentric Loading

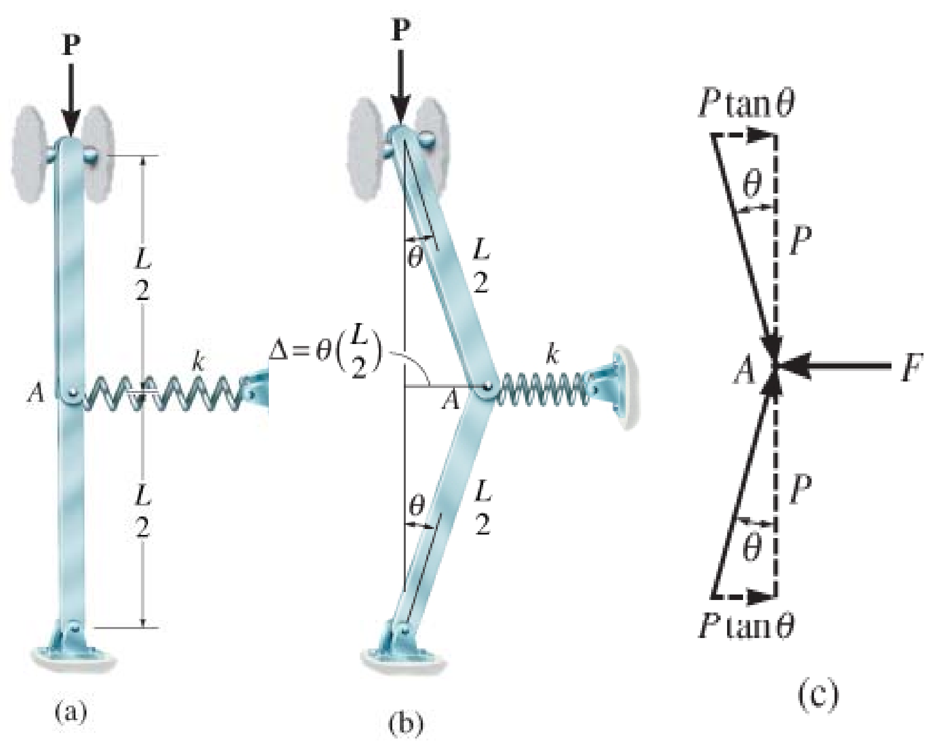
13.1 CRITICAL LOAD

- Long slender members subjected to axial compressive force are called columns.
- The lateral deflection that occurs is called buckling.
- The maximum axial load a column can support when it is on the verge of buckling is called the critical load, P_{cr}.



13.1 CRITICAL LOAD

- Spring develops restoring force $F = k\Delta$, while applied load **P** develops two horizontal components, $P_x = P \tan\theta$, which tends to push the pin further out of equilibrium.
- Since θ is small, $\Delta = \theta(L/2)$ and $\tan\theta \approx \theta$.
- Thus, restoring spring force becomes $F = k\theta L/2$, and disturbing force is $2P_x = 2P\theta$.



13.1 CRITICAL LOAD

- For $k\theta L/2 > 2P\theta$,

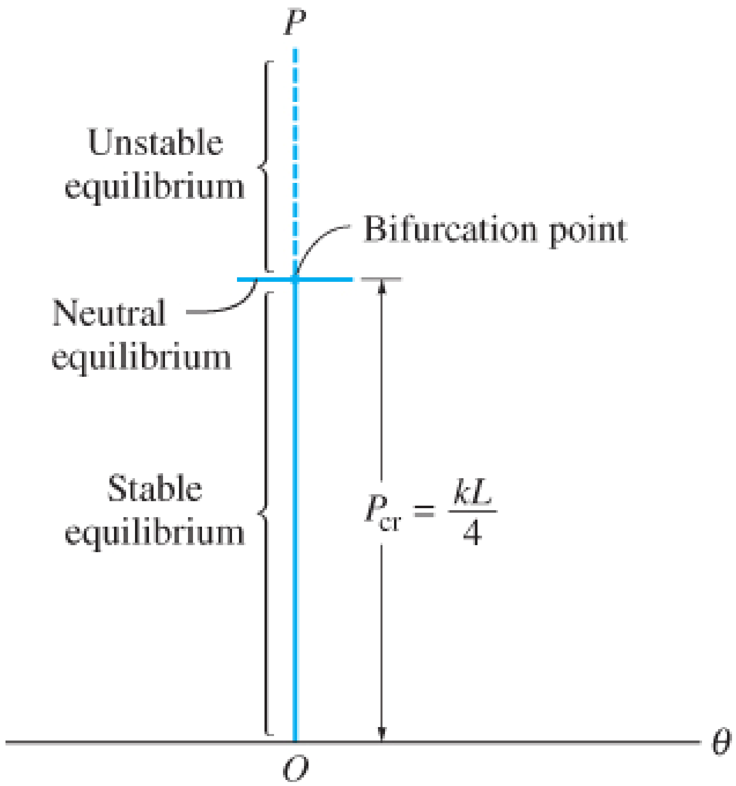
$$P < \frac{kL}{4}$$ stable equilibrium

- For $k\theta L/2 < 2P\theta$,

$$P > \frac{kL}{4}$$ unstable equilibrium

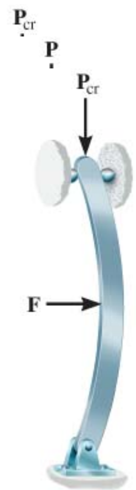
- For $k\theta L/2 = 2P\theta$,

$$P_{cr} = \frac{kL}{4}$$ neutral equilibrium



13.2 IDEAL COLUMN WITH PIN SUPPORTS

- An ideal column is perfectly straight before loading, made of homogeneous material, and upon which the load is applied through the centroid of the x-section.
- We also assume that the material behaves in a linear-elastic manner and the column buckles or bends in a single plane.



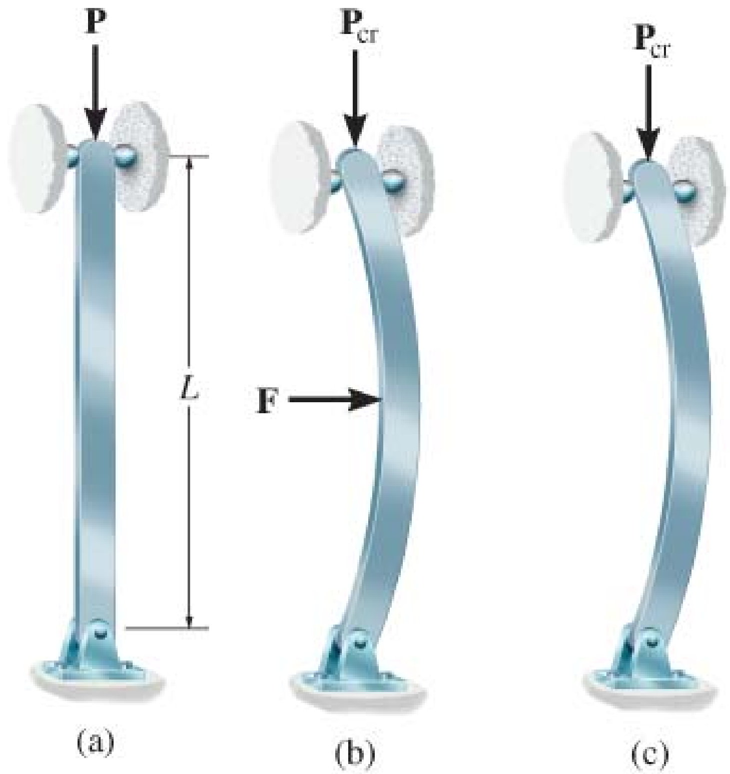
(b)

13.2 IDEAL COLUMN WITH PIN SUPPORTS

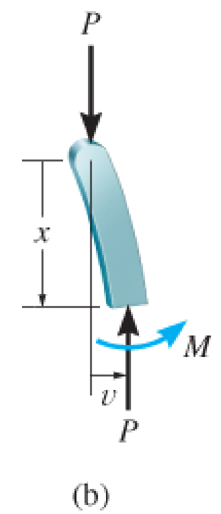
- In order to determine the critical load and buckled shape of column, we apply Eqn 12-10,

$$EI\frac{d^2\upsilon}{dx^2} = M \qquad (13\text{-}1)$$

- Recall that this eqn assume the slope of the elastic curve is small and deflections occur only in bending. We assume that the material behaves in a linear-elastic manner and the column buckles or bends in a single plane.



13.2 IDEAL COLUMN WITH PIN SUPPORTS



- Summing moments, $M = -P\nu$, Eqn 13-1 becomes

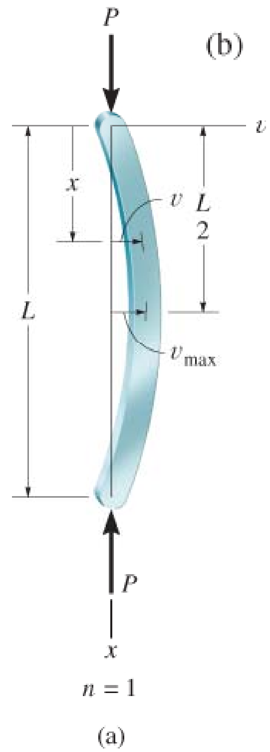
$$\frac{d^2\upsilon}{dx^2} + \left(\frac{P}{EI}\right)\upsilon = 0 \qquad (13\text{-}2)$$

- General solution is

$$\upsilon = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) \qquad (13\text{-}3)$$

- Since $\nu = 0$ at $x = 0$, then $C_2 = 0$. Since $\nu = 0$ at $x = L$, then

$$C_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$



13.2 IDEAL COLUMN WITH PIN SUPPORTS

- Disregarding trivial soln for C_1 = 0, we get

$$\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

- Which is satisfied if

$$\sqrt{\frac{P}{EI}}L = n\pi$$

- or

$$P = \frac{n^2\pi^2 EI}{L^2} \qquad n = 1,2,3,\ldots$$

13.2 IDEAL COLUMN WITH PIN SUPPORTS

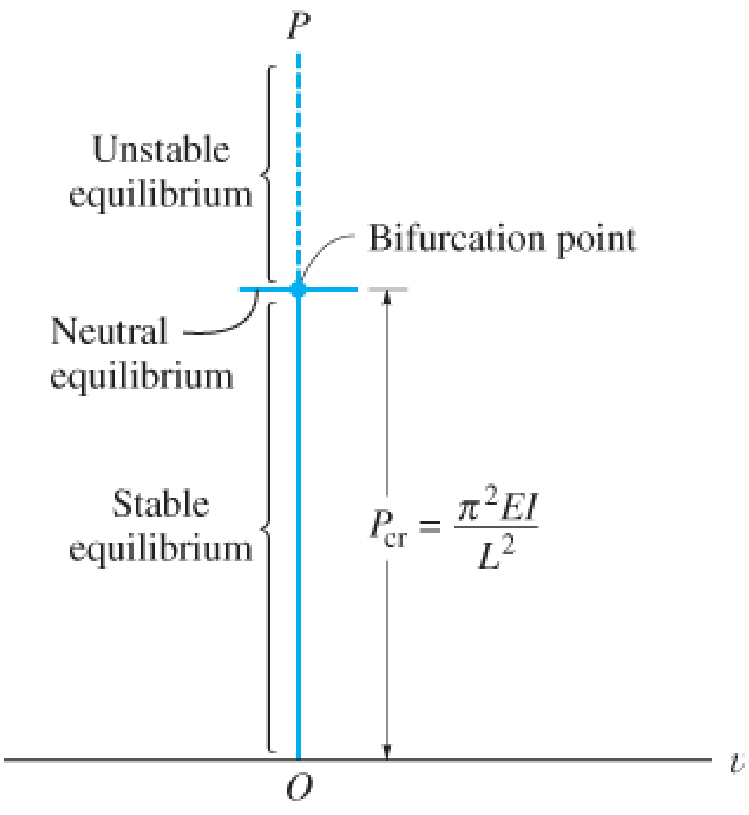
- Smallest value of P is obtained for n = 1, so critical load for column is

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

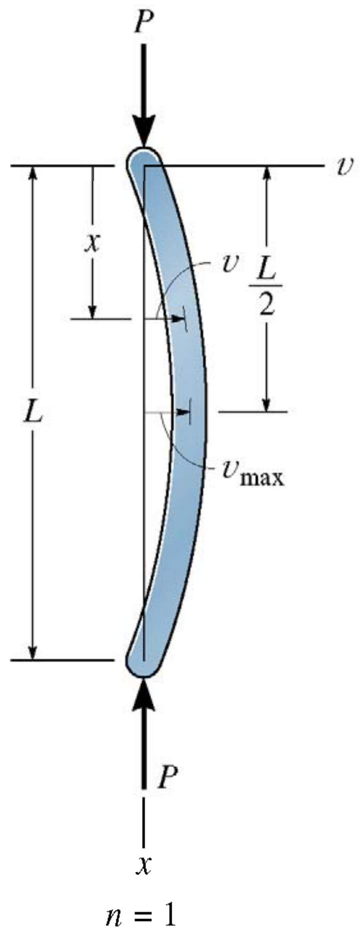
- This load is also referred to as the Euler load. The corresponding buckled shape is defined by

$$\upsilon = C_1 \sin \frac{\pi x}{L}$$

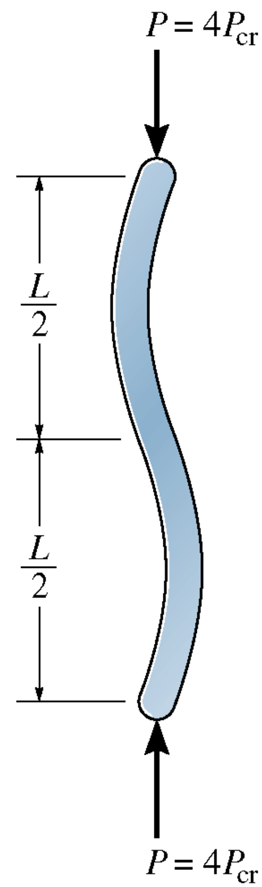
- C_1 represents maximum deflection, v_{max}, which occurs at midpoint of the column.



13.2 IDEAL COLUMN WITH PIN SUPPORTS



(a)



$n = 2$

(c)

13.2 IDEAL COLUMN WITH PIN SUPPORTS

- A column will buckle about the principal axis of the x-section having the least moment of inertia (weakest axis).
- For example, the meter stick shown will buckle about the a-a axis and not the b-b axis.
- Thus, circular tubes made excellent columns, and square tube or those shapes having $I_x \approx I_y$ are selected for columns.

13.2 IDEAL COLUMN WITH PIN SUPPORTS

- Buckling eqn for a pin-supported long slender column,

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad (13\text{-}5)$$

P_{cr} = critical or maximum axial load on column just before it begins to buckle. This load must not cause the stress in column to exceed proportional limit.

E = modulus of elasticity of material

I = Least modulus of inertia for column's x-sectional area.

L = unsupported length of pinned-end columns.

13.2 IDEAL COLUMN WITH PIN SUPPORTS

- Expressing $I = Ar^2$ where A is x-sectional area of column and r is the radius of gyration of x-sectional area.

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \qquad (13\text{-}6)$$

σ_{cr} = critical stress, an average stress in column just before the column buckles. This stress is an elastic stress and therefore $\sigma_{cr} \leq \sigma_Y$

E = modulus of elasticity of material

L = unsupported length of pinned-end columns.

r = smallest radius of gyration of column, determined from $r = \sqrt{(I/A)}$, where I is least moment of inertia of column's x-sectional area A.

13.2 IDEAL COLUMN WITH PIN SUPPORTS

- The geometric ratio L/r in Eqn 13-6 is known as the slenderness ratio.
- It is a measure of the column's flexibility and will be used to classify columns as long, intermediate or short.

13.2 IDEAL COLUMN WITH PIN SUPPORTS

IMPORTANT

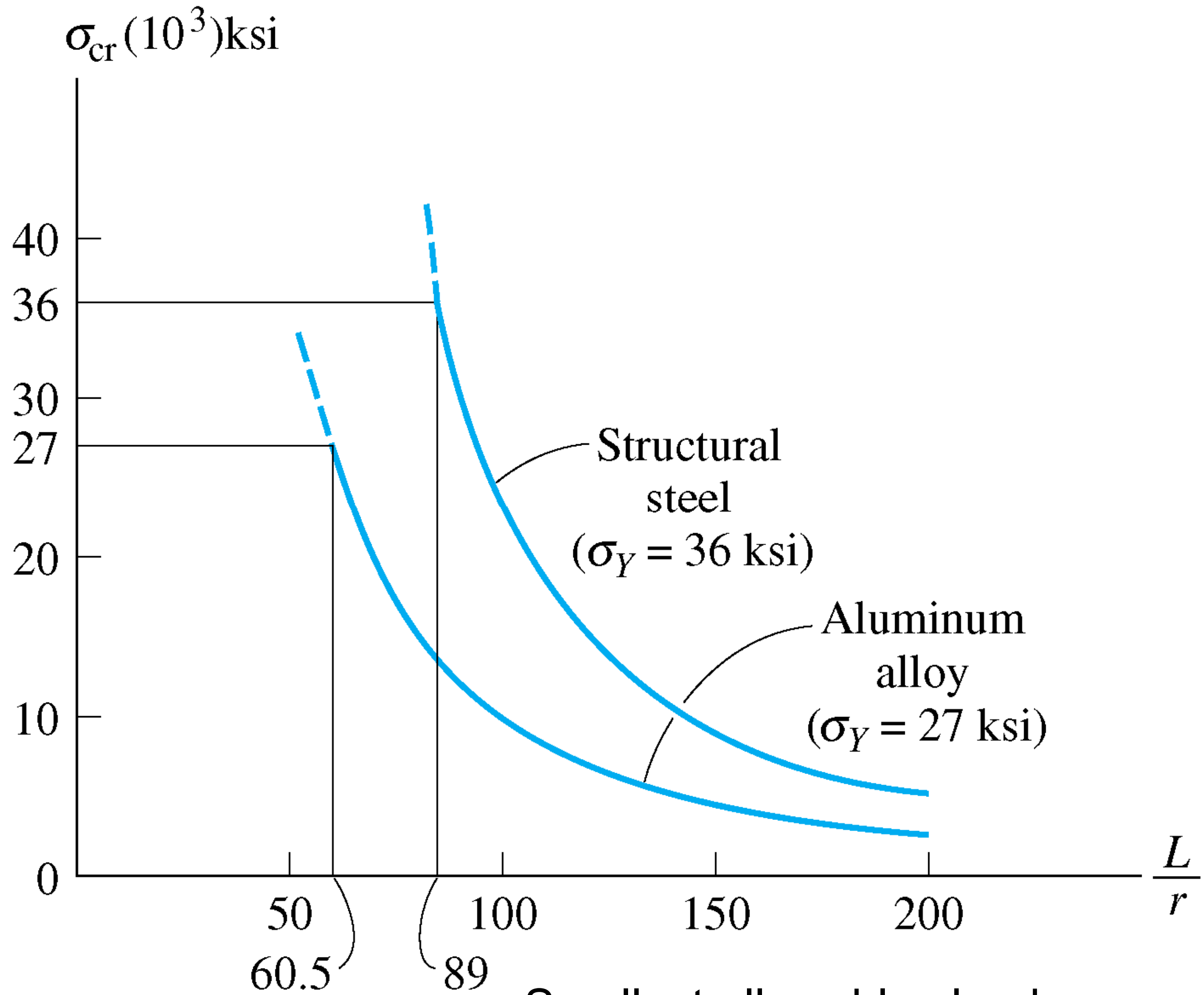
- Columns are long slender members that are subjected to axial loads.
- Critical load is the maximum axial load that a column can support when it is on the verge of buckling.
- This loading represents a case of neutral equilibrium.

13.2 IDEAL COLUMN WITH PIN SUPPORTS

IMPORTANT

- An ideal column is initially perfectly straight, made of homogeneous material, and the load is applied through the centroid of the x-section.
- A pin-connected column will buckle about the principal axis of the x-section having the least moment of intertia.
- The slenderness ratio L/r, where r is the smallest radius of gyration of x-section. Buckling will occur about the axis where this ratio gives the greatest value.

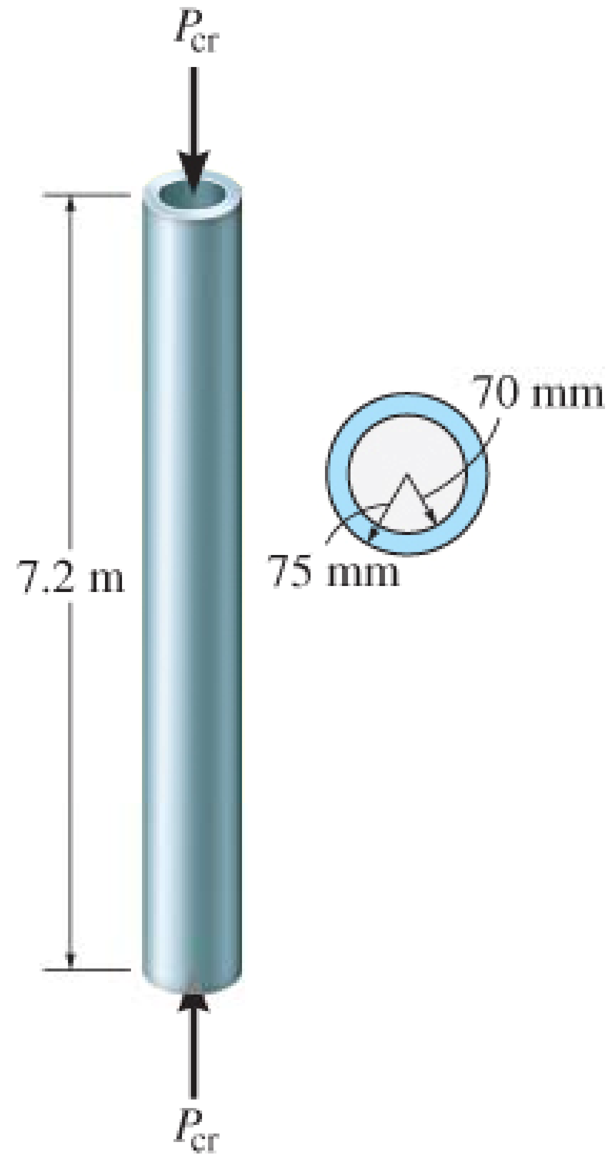
13.2 IDEAL COLUMN WITH PIN SUPPORTS



→ Smallest allowable slenderness ratio

EXAMPLE (6th Ed.)

A 7.2-m long A-36 steel tube having the x-section shown is to be used a pin-ended column. Determine the maximum allowable axial load the column can support so that it does not buckle.



Pcr = Pi^2EI/L^2

EXAMPLE (SOLN)

Use Eqn 13-5 to obtain critical load with E_{st} = 200 GPa.

$$P_{cr} = \frac{\pi^2 EI}{L_2}$$

$$= \frac{\pi^2 \left[200\left(10^6\right) \text{kN/m}^2\right] - \frac{1}{4}\pi(70)^4 (1\text{ m}/1000\text{ mm})^4}{(7.2\text{ m})^2}$$

$$= 228.2\text{ kN}$$

EXAMPLE (SOLN)

This force creates an average compressive stress in the column of

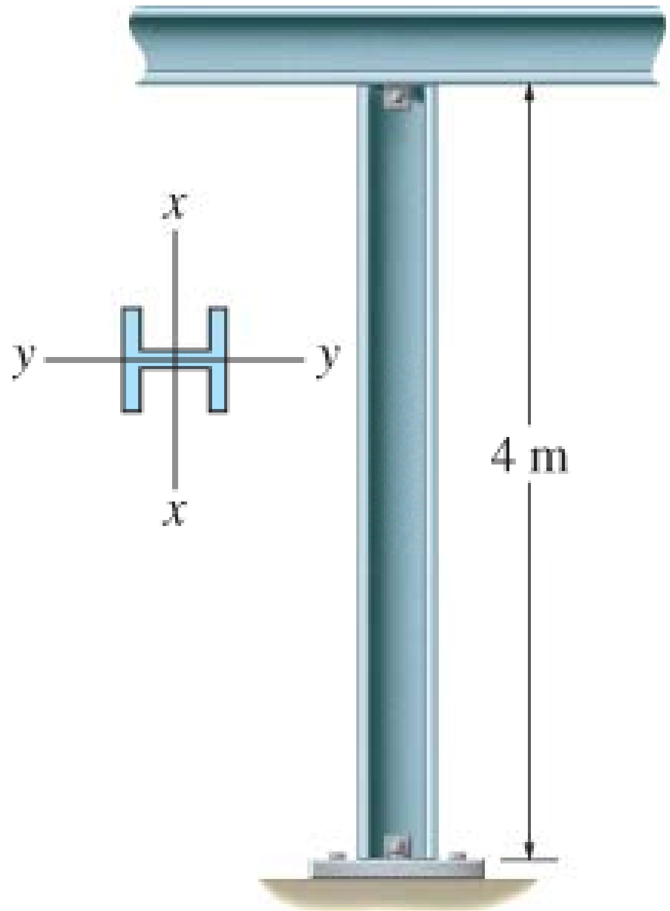
$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{228.2\ \text{kN}\left(1000\ \text{N/kN}\right)}{\left[\pi(75)^2 - \pi(70)^2\right]\text{mm}^2}$$

$$= 100.2\ \text{N/mm}^2 = 100\ \text{MPa}$$

Since $\sigma_{cr} < \sigma_Y$ = 250 MPa, application of Euler's eqn is appropriate.

EXAMPLE 13.1

The A-36 steel W200×46 member shown is to be used as a pin-connected column. Determine the largest axial load it can support before it either begins to buckle or the steel yields.



Pcr = Pi^2EI/L^2

σ = Pcr/A

EXAMPLE 13.1 (SOLN)

From table in Appendix B, column's x-sectional area and moments of inertia are A = 5890 mm^2, I_x = 45.5×10^6 mm^4,and I_y = 15.3×10^6 mm^4.

By inspection, buckling will occur about the y-y axis.

Applying Eqn 13-5, we have

$$P_{cr} = \frac{\pi^2 EI}{L_2}$$

$$= \frac{\pi^2\left[200\left(10^6\right)\text{kN/m}^2\right]\left(15.3\left(10^4\right)\text{mm}^4\right)\left(1\text{ m}/1000\text{ mm}\right)^4}{\left(4\text{ m}\right)^2}$$

$$= 1887.6\text{ kN}$$

EXAMPLE 13.1 (SOLN)

When fully loaded, average compressive stress in column is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{1887.6\ \text{kN}(1000\ \text{N/kN})}{5890\ \text{mm}^2}$$

$$= 320.5\ \text{N/mm}^2$$

Since this stress exceeds yield stress (250 N/mm^2), the load P is determined from simple compression:

$$250\ \text{N/mm}^2 = \frac{P}{5890\ \text{mm}^2}$$

$$P = 1472.5\ \text{kN}$$

COLUMNS HAVING VARIOUS TYPES OF SUPPORTS

$$EI\frac{d^2v}{dx^2} = P(\delta - v)$$

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = \frac{P}{EI}\delta \qquad (13\text{–}7)$$

Unlike Eq. 13–2, this equation is nonhomogeneous because of the nonzero term on the right side. The solution consists of both a complementary and a particular solution, namely,

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) + \delta$$

The constants are determined from the boundary conditions. At $x = 0$, $v = 0$, so that $C_2 = -\delta$. Also,

$$\frac{dv}{dx} = C_1\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}x\right) - C_2\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}x\right)$$

At $x = 0$, $dv/dx = 0$, so that $C_1 = 0$. The deflection curve is therefore

$$v = \delta\left[1 - \cos\left(\sqrt{\frac{P}{EI}}x\right)\right] \qquad (13\text{–}8)$$

Since the deflection at the top of the column is δ, that is, at $x = L$, $v = \delta$, we require

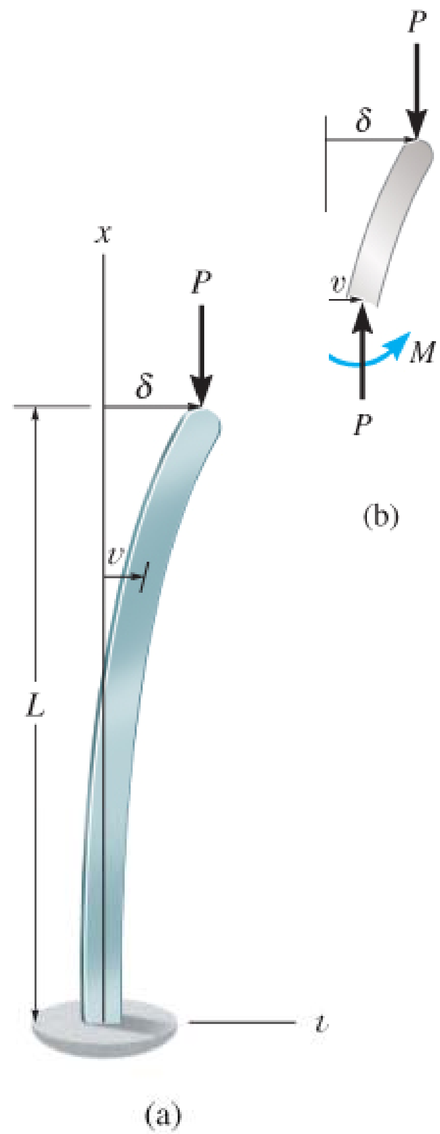
$$\delta \cos\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

The trivial solution $\delta = 0$ indicates that no buckling occurs, regardless of the load P. Instead,

$$\cos\left(\sqrt{\frac{P}{EI}}L\right) = 0 \quad \text{or} \quad \sqrt{\frac{P}{EI}}L = \frac{n\pi}{2}, n = 1, 3, 5 \ldots$$

The smallest critical load occurs when $n = 1$, so that

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \qquad (13\text{–}9)$$



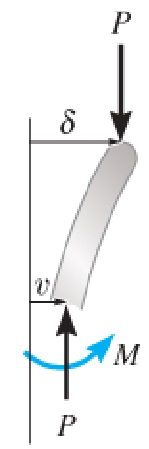
13.3 COLUMNS HAVING VARIOUS TYPES OF SUPPORTS

- From free-body diagram, $M = P(\delta - v)$.
- Differential eqn for the deflection curve is

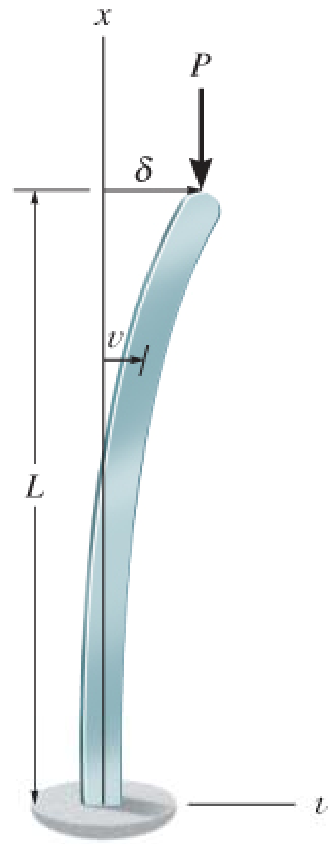
$$\frac{d^2\upsilon}{dx^2} + \frac{P}{EI}\upsilon = \frac{P}{EI}\delta \qquad (13\text{-}7)$$

- Solving by using boundary conditions and integration, we get

$$\upsilon = \delta\left[1 - \cos\left(\sqrt{\frac{P}{EI}}x\right)\right] \qquad (13\text{-}8)$$



(b)



(a)

13.3 COLUMNS HAVING VARIOUS TYPES OF SUPPORTS

- Thus, smallest critical load occurs when n = 1, so that

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \qquad (13\text{-}9)$$

- By comparing with Eqn 13-5, a column fixed-supported at its base will carry only one-fourth the critical load applied to a pin-supported column.

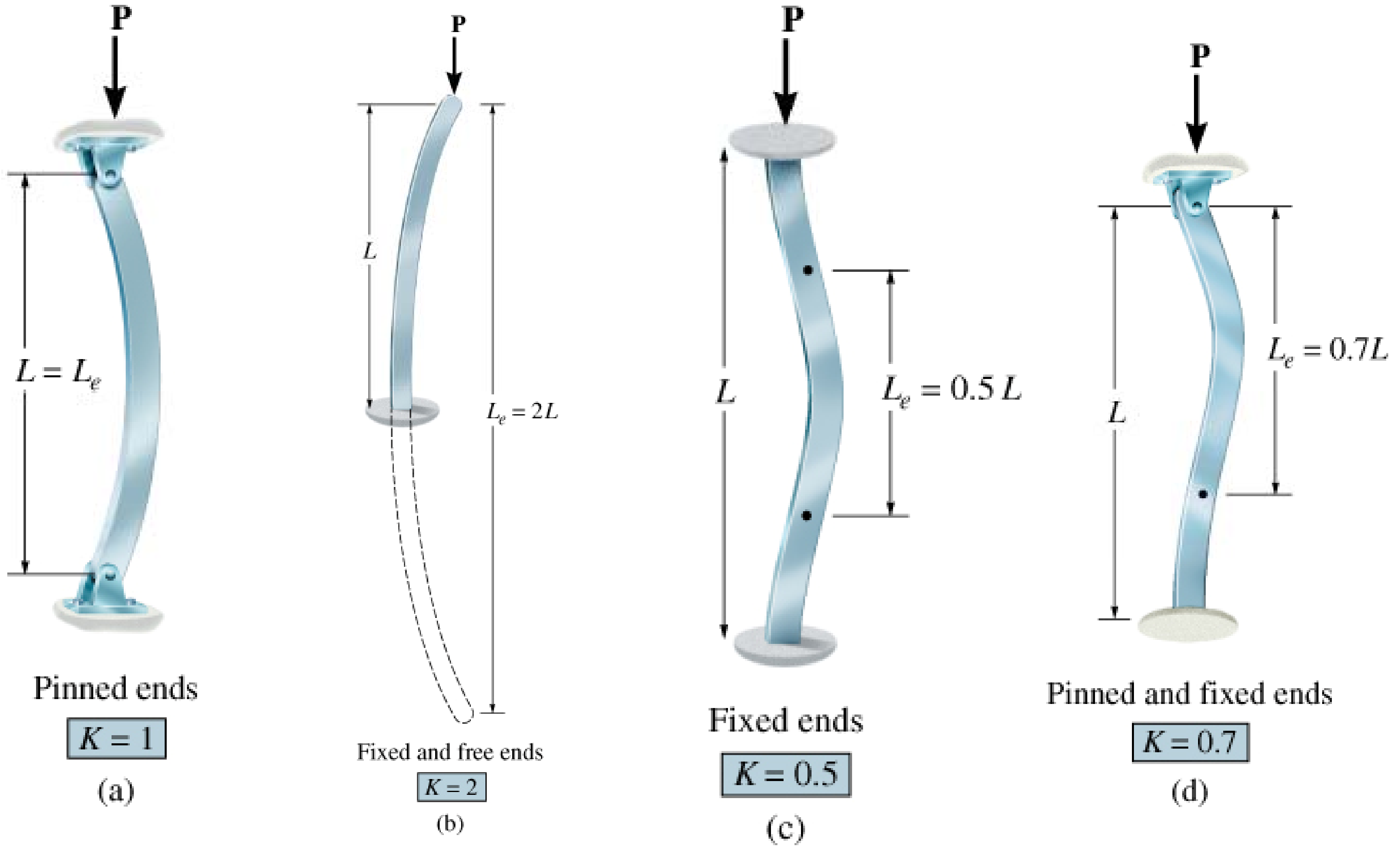
13.3 COLUMNS HAVING VARIOUS TYPES OF SUPPORTS

Effective length

- If a column is not supported by pinned-ends, then Euler's formula can also be used to determine the critical load.
- "L" must then represent the distance between the zero-moment points.
- This distance is called the columns' effective length, L_e.

13.3 COLUMNS HAVING VARIOUS TYPES OF SUPPORTS

Effective length



13.3 COLUMNS HAVING VARIOUS TYPES OF SUPPORTS

Effective length

- Many design codes provide column formulae that use a dimensionless coefficient K, known as the effective-length factor.

$$L_e = KL \qquad (13\text{-}10)$$

- Thus, Euler's formula can be expressed as

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} \qquad (13\text{-}11)$$

$$\sigma_{cr} = \frac{\pi^2 E}{(KL/r)^2} \qquad (13\text{-}12)$$

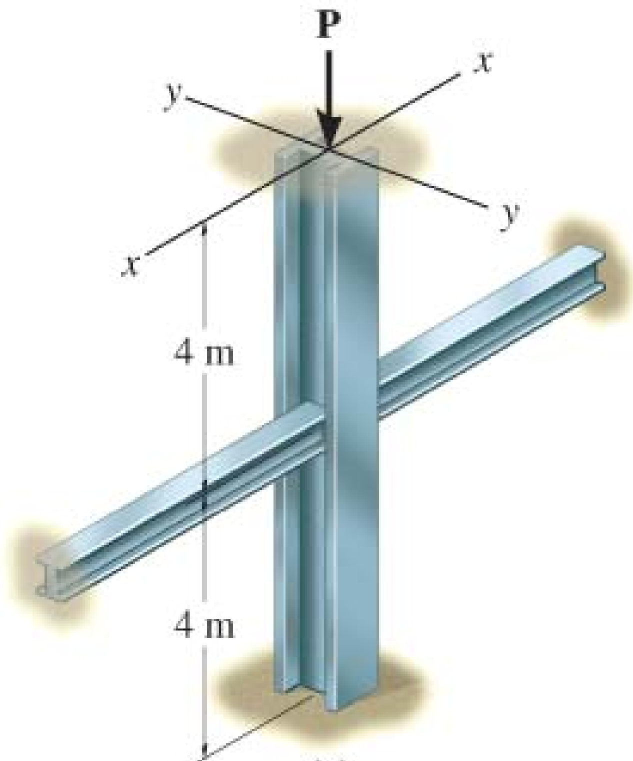
13.3 COLUMNS HAVING VARIOUS TYPES OF SUPPORTS

Effective length

- Here (KL/r) is the column's effective-slenderness ratio.

EXAMPLE 13.2

A W150×24 steel column is 8 m long and is fixed at its ends as shown. Its load-carrying capacity is increased by bracing it about the y-y axis using struts that are assumed to be pin-connected to its mid-height. Determine the load it can support so that the column does not buckle nor material exceed the yield stress.



(a)

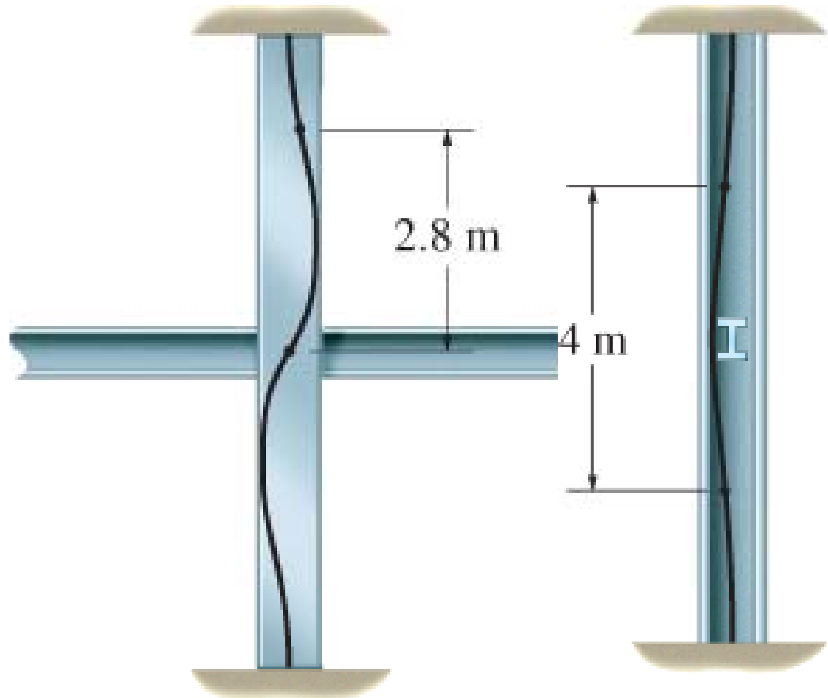
Take E_{st} = 200 GPa and σ_Y = 410 MPa.

EXAMPLE 13.2 (SOLN)

Buckling behavior is different about the x and y axes due to bracing.

Buckled shape for each case is shown.

The effective length for buckling about the x-x axis is $(KL)_x = 0.5(8\text{ m}) = 4\text{ m}$.



y–y axis buckling

(c)

x–x axis buckling

(b)

For buckling about the y-y axis, $(KL)_y = 0.7(8\text{ m}/2) = 2.8\text{ m}$.

We get $I_x = 13.4\times10^6\text{ mm}^4$ and $I_y = 1.83\times10^6\text{ mm}^4$ from Appendix B.

EXAMPLE 13.2 (SOLN)

Applying Eqn 13-11,

$$(P_{cr})_x = \frac{\pi^2 EI_x}{(KL)_x^2} = \frac{\pi^2 \left[200\left(10^6\right)\text{kN/m}^2\right]13.4\left(10^{-6}\right)\text{m}^4}{(4\text{ m})^2}$$

$$(P_{cr})_x = 1653.2\ \text{kN}$$

$$(P_{cr})_y = \frac{\pi^2 EI_y}{(KL)_y^2} = \frac{\pi^2 \left[200\left(10^6\right)\text{kN/m}^2\right]1.83\left(10^{-6}\right)\text{m}^4}{(2.8\text{ m})^2}$$

$$(P_{cr})_y = 460.8\ \text{kN}$$

By comparison, buckling will occur about the y-y axis.

EXAMPLE 13.2 (SOLN)

Area of x-section is 3060 mm^2, so average compressive stress in column will be

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{460.8\left(10^3\right)\text{ N}}{3060\text{ m}^2} = 150.6\text{ N/mm}^2$$

Since $\sigma_{\text{cr}} < \sigma_Y$ = 410 MPa, buckling will occur before the material yields.

EXAMPLE 13.2 (SOLN)

NOTE: From Eqn 13-11, we see that buckling always occur about the column axis having the largest slenderness ratio. Thus using data for the radius of gyration from table in Appendix B,

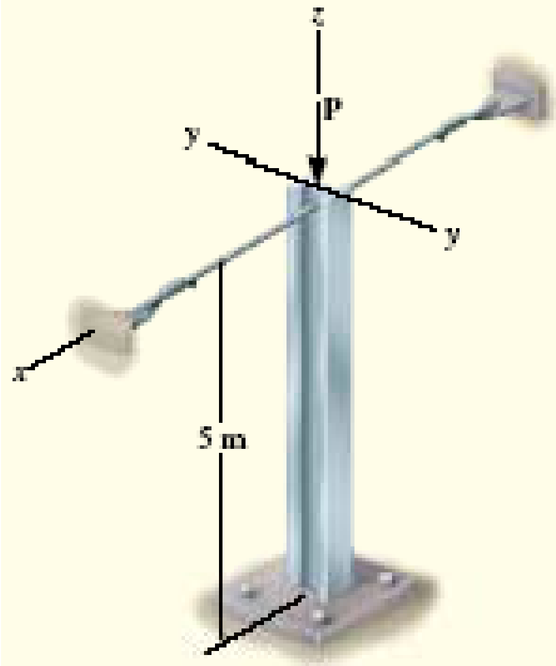
$$\left(\frac{KL}{r}\right)_x = \frac{4\ \text{m}(1000\ \text{mm/m})}{66.2\ \text{mm}} = 60.4$$

$$\left(\frac{KL}{r}\right)_y = \frac{2.8\ \text{m}(1000\ \text{mm/m})}{24.5\ \text{mm}} = 114.3$$

Hence, y-y axis buckling will occur, which is the same conclusion reached by comparing Eqns 13-11 for both axes.

EXAMPLE 13.3 (SOLN)

The aluminum column is fixed at its bottom and is braced at its top by cables so as to prevent movement at the top along the x axis, Fig. 13–14a. If it is assumed to be fixed at its base, determine the largest allowable load P that can be applied. Use a factor of safety for buckling of F.S. = 3.0. Take $E_{al} = 70$ GPa, $\sigma_Y = 215$ MPa, $A = 7.5(10^{-3})\ \text{m}^2$, $I_x = 61.3(10^{-6})\ \text{m}^4$, $I_y = 23.2(10^{-6})\ \text{m}^4$.



(a)

Solution

Buckling about the x and y axes is shown in Fig. 13–14b and 13–14c, respectively. Using Fig. 13–12a, for x–x axis buckling, $K = 2$, so $(KL)_x = 2(5\text{ m}) = 10\text{ m}$. Also, for y–y axis buckling, $K = 0.7$, so $(KL)_y = 0.7(5\text{ m}) = 3.5\text{ m}$.

Applying Eq. 13–11, the critical loads for each case are

$$(P_{cr})_x = \frac{\pi^2 EI_x}{(KL)_x^2} = \frac{\pi^2[70(10^9)\text{ N/m}^2](61.3(10^{-6})\text{ m}^4)}{(10\text{ m})^2}$$
$$= 424\text{ kN}$$

$$(P_{cr})_y = \frac{\pi^2 EI_y}{(KL)_y^2} = \frac{\pi^2[70(10^9)\text{ N/m}^2](23.2(10^{-6})\text{ m}^4)}{(3.5\text{ m})^2}$$
$$= 1.31\text{ MN}$$

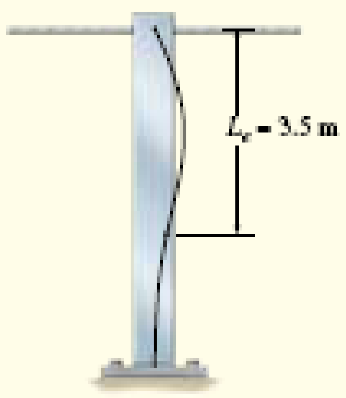
By comparison, as P is increased the column will buckle about the x–x axis. The allowable load is therefore

$$P_{allow} = \frac{P_{cr}}{\text{F.S.}} = \frac{424\text{ kN}}{3.0} = 141\text{ kN} \qquad \textit{Ans.}$$

Since

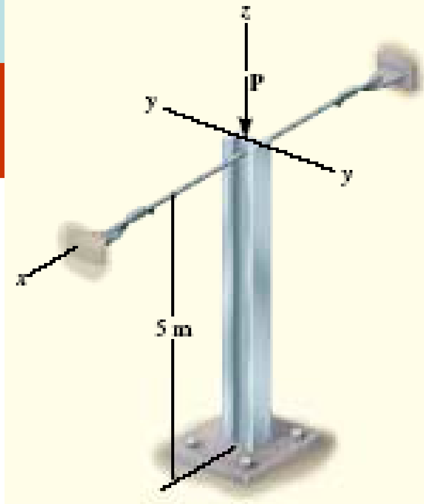
$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{424\text{ kN}}{7.5(10^{-3})\text{ m}^2} = 56.5\text{ MPa} < 215\text{ MPa}$$

Euler's equation can be applied.

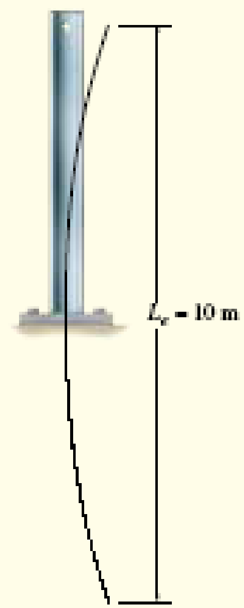


y–y axis buckling

(c)



(a)



x—x axis buckling

(b)