King Saud University

College of Science

Department of Mathematics

151 Math   Exercises

(5.2)

Graph Terminology and Special Types of Graphs

Malek Zein AL–Abidin

ﻫ1440
_____
2018

Basic Terminology

First, we give some terminology that describes the vertices and edges of undirected graphs.

DEFINITION 1 Two vertices u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G if u and v are endpoints of an edge e of G. Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v.

DEFINITION 2 The set of all neighbors of a vertex v of $G = (V,E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \_v{\in}A \, N(v)$.

DEFINITION 3 The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

EXAMPLE 1 What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure 1?

Solution: In G, $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, and $\deg(g) = 0$.

The neighborhoods of these vertices are $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$.

In H, $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, and $\deg(d) = 5$. The neighborhoods of these vertices are $N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$.
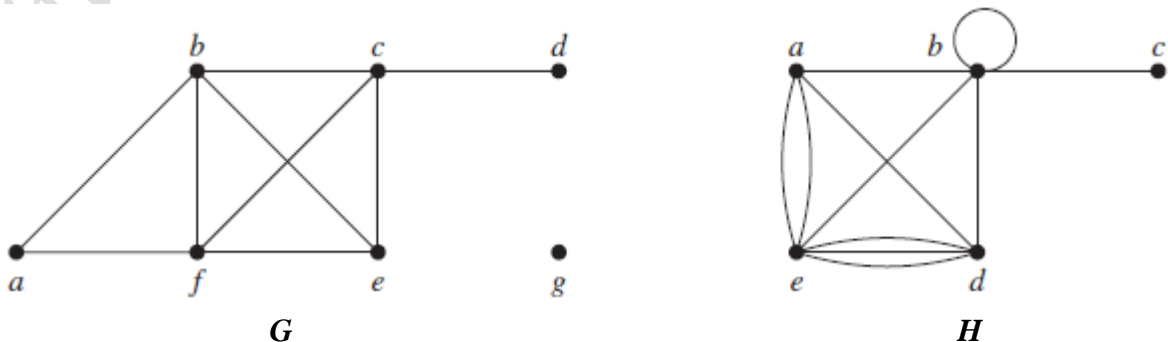

FIGURE 1 The Undirected Graphs G and H.

THEOREM 1

THE HANDSHAKING THEOREM Let $G = (V, E)$ be an undirected graph with m edges $m = |E|$. Then

$$2m = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

EXAMPLE 3 How many edges are there in a graph with 10 vertices each of degree six?
Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$.

THEOREM 2 An undirected graph has an even number of vertices of odd degree.

DEFINITION 4 When *(u, v)* is an edge of the graph G with directed edges, u is said to be *adjacent to v* and v is said to be *adjacent from u*. The vertex u is called the *initial vertex* of *(u, v)*, and v is called the *terminal* or *end vertex* of *(u, v)*. The initial vertex and terminal vertex of a loop are the same.

DEFINITION 5 In a graph with directed edges the *in-degree of a vertex v*, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The *out-degree of v*, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

EXAMPLE 4 Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in Figure 2.
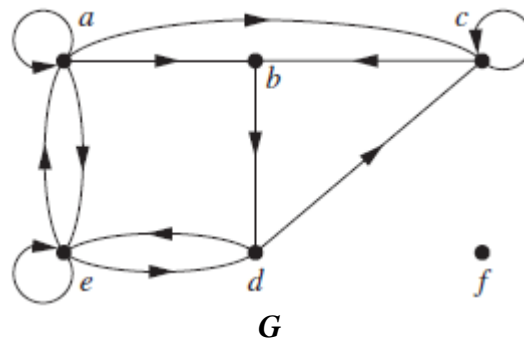

G
FIGURE 2 The Directed Graph G.

Solution: The in-degrees in G are $\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$, $\deg^-(e) = 3$, and $\deg^-(f) = 0$. The out-degrees are $\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$, $\deg^+(e) = 3$, and $\deg^+(f) = 0$.

THEOREM 3   Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|$$

Some Special Simple Graphs

EXAMPLE 5 Complete Graphs A **complete graph on *n* vertices**, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. The graphs K_n for $n = $ 1, 2, 3, 4, 5, 6, are displayed in Figure 3. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete** .

K_1 K_2 K_3 K_4 K_5 K_6
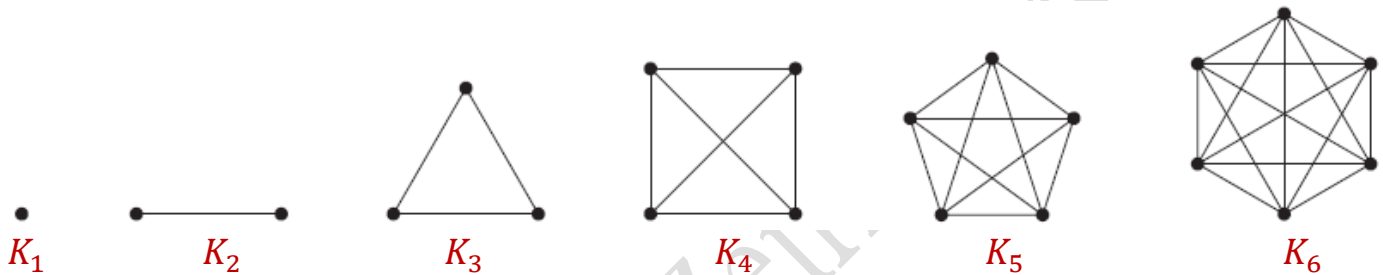
FIGURE 3 The Graphs K_n for $1 \leq n \leq 6$.

EXAMPLE 6 Cycles A **cycle** C_n , $n \geq 3$, consists of *n* vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4.
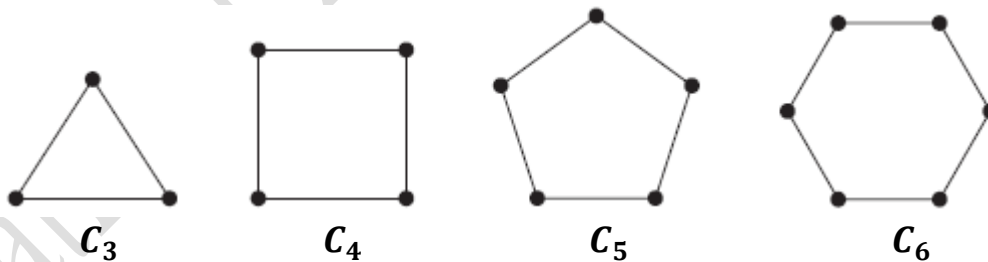

C_3 C_4 C_5 C_6

FIGURE 4 The Cycles C_3, C_4, C_5, and C_6 .

EXAMPLE 7 Wheels We obtain a **wheel** W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the *n* vertices in C_n, by new edges. The wheels W_3, W_4, W_5 and W_6 are displayed in Figure 5.
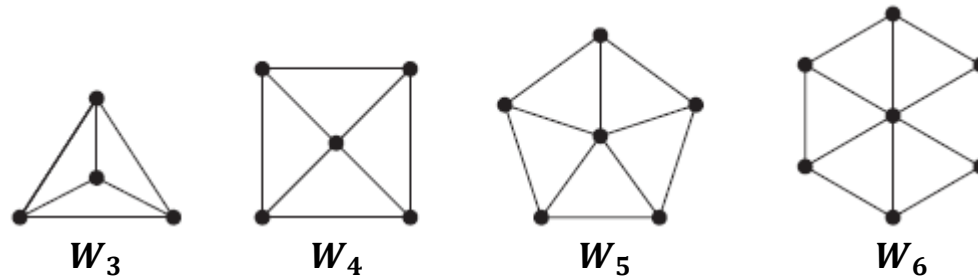
$$W_3 \qquad W_4 \qquad W_5 \qquad W_6$$

FIGURE 5 The Wheels W_3, W_4, W_5 and W_6

Bipartite Graphs

DEFINITION 6 A simple graph G is called ***bipartite*** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.

THEOREM 4 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

EXAMPLE 13 Complete Bipartite Graphs A **complete bipartite graph** $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset. The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$, and $K_{2,6}$ are displayed in Figure 9.

$$K_{2,3} \qquad\qquad K_{3,3}$$

$$K_{3,5} \qquad\qquad K_{2,6}$$
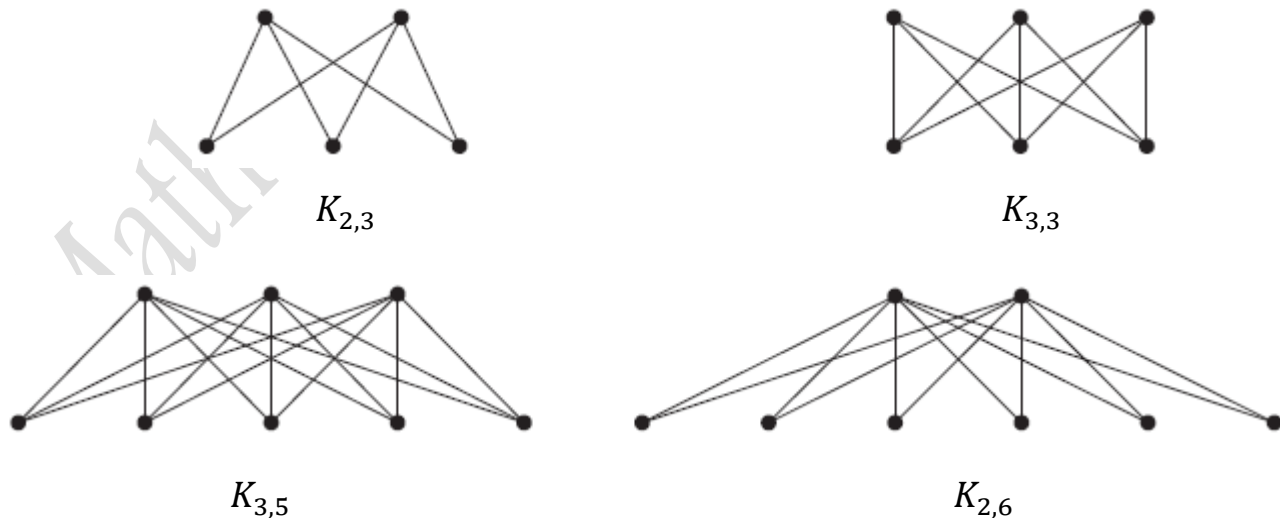
DEFINITION 7 A *subgraph of a graph G = (V ,E)* is a graph *H = (W, F)*, where $W \subseteq V$ and $F \subseteq E$.

A subgraph *H* of *G* is a *proper subgraph* of *G* if $H \ne G$.

DEFINITION 8 Let *G = (V ,E)* be a simple graph. The **subgraph induced** by a subset *W* of the vertex set *V* is the graph (*W, F*), where the edge set *F* contains an edge in *E* if and only if both endpoints of this edge are in *W*.

EXAMPLE 18 The graph *G* shown in Figure 15 is a subgraph of K_5 . If we add the edge connecting *c* and *e* to *G*, we obtain the subgraph induced by $W = \{a, b, c, e\}$.
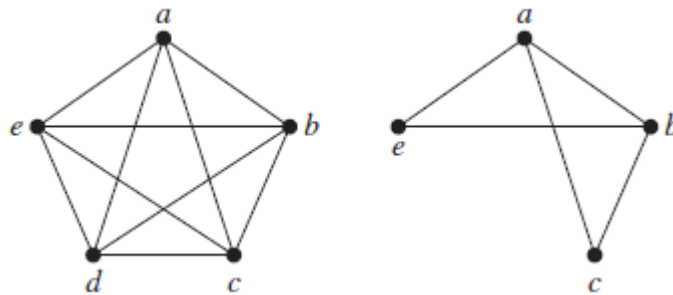
FIGURE 15 A Subgraph of K_5 **.**

DEFINITION 9 A simple graph is called **regular** if every vertex of this graph has the same degree.

A regular graph is called ***r*-regular** if every vertex in this graph has degree ***r***.

THEOREM 5 If $K_n(V, E)$ then $|E| = \dfrac{n(n-1)}{2}$

THEOREM 6 If $K_{m,n}(V_1 \cup V_2, E)$ where $|V_1| = m$, $|V_2| = n \Rightarrow |E| = mn$
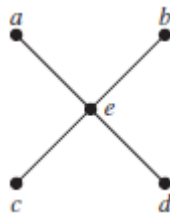
Exercises

1. Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices. Find the sum of the degrees of the vertices of the graph and verify that it equals twice the number of edges in the graph.
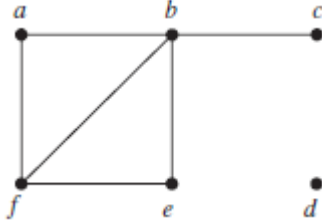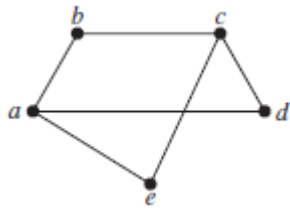

2. In Exercises 21–25 determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.
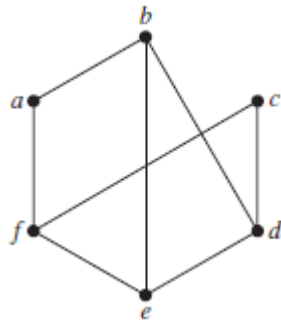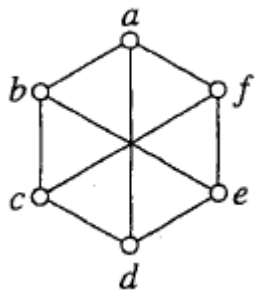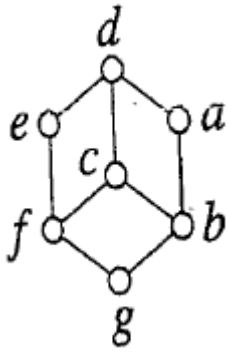
(a)

(b)

(c)

(d)

(e)

(f)

(g)