First Semester 1432/1433 H
Final Examination
5811 Math
Duration: 3 Hours

First Question :

(i) State and prove Baire's Category Theorem.

(ii) Let X be a normed space such that absolute convergence of any series always implies convergence of that series. Prove that X is complete.

Second Question :

Prove or disprove each of the following:

(1) There is a non-reflexive Banach space X whose dual X^* is reflexive.

(2) For each x in a normed space X, $\|x\| = \sup_{f \in X^* \setminus \{0\}} \frac{|f(x)|}{\|f\|}$.

(3) If X is a Banach space, Y is a normed space and $T_n \in B(X,Y)$ such that $(T_n(x))_{n=1}^{\infty}$ is a Cauchy sequence in Y for every $x \in X$, then $(\|T_n)\|)_{n=1}^{\infty}$ is bounded.

(4) Any closed linear operator $T : X \to Y$ of normed spaces X and Y is bounded.

(5) If $f \neq 0$ is a linear functional on a normed space X, then $f \in X^*$.

Third Question :

(i) Let $M \neq \phi$ be a closed convex subset of a Hilbert space H. Prove that M contains a unique vector of smallest norm.

(ii) Let X and Y be normed spaces, and let $T : X \to Y$ be a a closed linear operator. Prove that:

(1) The null space $N(T)$ is a closed subspace of X.

(2) If Y is compact, then T is bounded.

(3) If X is compact, and T is bijective, then T^{-1} is bounded.

Forth Question :

(i) Let $T : l^{\infty} \to l^{\infty}$ be the linear map defined by $T((\xi_i)) = (\frac{\xi_i}{i})$. Show that T is bounded. Is T an open map?. Is T a colsed map?. Justify your answers.

(ii) Let $A = (\alpha_{jk})$ be an $r \times n$ matrix of real numbers. Show that A defines a bounded linear operator $A : (\mathbb{R}^n, \|.\|_1) \to (\mathbb{R}^r, \|.\|_2)$, where $\|(\xi_1, ..., \xi_n)\|_1 =$

$\sum_{k=1}^{n} |\xi_k|$, and $\|(\eta_1, ..., \eta_r)\|_2 = \sum_{j=1}^{r} |\eta_j|$. Also, prove that the norm $\|A\|$ of A given by $\|A\| = \max_k \sum_{i=1}^{r} |\alpha_{jk}|$ is compatable with $\|.\|_1$ and $\|.\|_2$.

(iii) Let Y and Z be closed subspaces of a Hilbert space H such that $Y \perp Z$. Prove that the subspace $Y + Z$ is also closed.

Fifth Question :

(i) Let $T : X \to Y$ be a bounded linear operator of normed spaces X and Y. Prove the following:

(1) There is a bounded linear oprator $T^{\times} : Y^* \to X^*$ defined by $(T^{\times}(g))(x) = g(Tx)$ for all $x \in X$, $g \in Y^*$, and $\left\|T^{\times}\right\| = \|T\|$.

(2) If T^{-1} exists and bounded,then $(T^{\times})^{-1}$ exists, bounded and $(T^{\times})^{-1} = (T^{-1})^{\times}$.

(3) If X and Y are Hilbert spaces, what is the relation between $T^{\times}$ and the Hilbert adjoint operator T^*?.

First Semester 1429/1430 H
Final Examination
581 Math
Duration: 3 Hours

First Question :

(i) Let X and Y be normed spaces and let $T : X \to Y$ be a non-zero linear operator such that T continuous at a point $x_\circ \in X$. Show that:

(1) T is bounded on X.

(2) $\|Tx\| < \|T\|$ for all $x \in X$ such that $\|x\| < 1$.

(ii) Let $f : X \to \mathbb{C}$ be a bounded non-zero linear functional, and let $Y = \{x \in X : f(x) = 1\}$. Prove that $\|f\| = \frac{1}{d}$, where d is the distance from Y to the origin.

Second Question :

(i) Let X and Y be normed spaces. Prove that if the space $B(X, Y)$ of all bounded linear oprators from X into Y is complete then Y is complete.

(ii) Consider the space $\mathbb{R}^n$ with the norm

$$\|x\| \max_{1 \le i \le n} |x_i|, \qquad x = (x_1, ..., x_n).$$

Find the dual $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$ with this norm.

(iii) Prove that any Hilbert space H is isometrically isomorphic to its dual H^*.

Third Question :

Prove or disprove each of the following:

(1) The normed space $(C[-1, 1], \|.\|)$ with the the norm defind by $\|x\| \max_{t \in [-1,1]} |x(t)|$ is a Hilbert space.

(2) Every finite dimensional normed space is reflexive.

(3) every bounded linear operator $T : D(T) \to Y$ is closed, where $D(T) \subseteq X$, X and Y are normed spaces.

(4) If Y is a subspace of a Hilbert space H such that $Y = Y^{\perp\perp}$, then Y is closed in H.

(5) If $X \ne \{0\}$ is a normed space, then its dual $X^* \ne \{0\}$.

Forth Question :

(i) Let T be a non-zero bounded linear operator of a normed space X onto a Banch space Y. Prove that, for each $n \in \mathbb{N}$, the norm closure $\overline{T(B_n)}$ of $T(B_n)$ contains an open ball about 0_Y, where $B_n = B(0_X; 2^{-n}) = \{x \in X : \|x\| < 2^{-n}\}$.

(ii) Let X be a subspace of l^∞ consists of all elements $x = (\xi_i)_{i=1}^\infty,\ \xi_i = 0$ for all but finite number of $i's$. Define $T : X \to X$ by $T(x) = (\frac{\xi_i}{i})_{i=1}^\infty$, $x = (\xi_i)_{i=1}^\infty$.

(1) Show that T is linear and bounded.
(2) Does $T^{-1} : R(T) \to X$ exists?.
(3) If T^{-1} exists, is it bounded?.

Fifth Question :

Let X and Y be Banach spaces, and let $T : D(T) \to Y$, be a closed linear operator, where $D(T) \subseteq X$. Prove that:

(1) If $D(T)$ is closed in X, then T is bounded.

(2) If $T^{-1} : R(T) \to X$ exists and is bounded, then $R(T)$ is closed in Y.

(3) If $T_n \in B(X, Y)$ such that $(T_n x)_{n=1}^\infty$ is a Cauchy sequence in Y, for every $x \in X$, then $(\|T_n\|)_{n=1}^\infty$ is bounded.

First Semester 1424/1425 H
The Final Examination
581 Math
Duration: 3 Hours

First Question

(i) : Define a C*-algebra, then show that the set $C(X)$ of all continuous complex valued functions on a compact set X is a commutative C*-algera with the norm given by $\|f\| = \sup_{x \in X} |f(x)|$.

(ii) : Let $\phi : A \to B$ be a *-homomorphism of C*-algebras A and B. Show that

(a) ϕ is continuous.
(b) $\sigma(\phi(x)) \subseteq \sigma(x)$ for each $x \in A$.

Second Question

Prove or disprove each of the following where A is a C*-algebra with identity I and B is a C*-subalgebra of A**:**

(1) If $x \in B$, then $\sigma_B(x) = \sigma_A(x)$.
(2) If $x \in B$ which is invertible in A, then $x^{-1} \in B$.
(3) If x is a normal element in A, then $r(x) < \|x\|$.
(4) If x is a self-adjoint element in A such that $\|x\| \geq 1$, then $\|I - x\| \geq 1$.
(5) There exists a non-zero element $x \in A$ such that $\rho(x) = 0$ for every state ρ of A.
(6) If $x, y \in A_{s.a}$ such that $-y \leq x \leq y$, then $\|x\| \leq \|y\|$.

Third Question

(i) Let H be a complex Hilbert space. Show that for each $\xi, \eta \in H$, the map $\omega_{\xi,\mu} : B(H) \to \mathbb{C}$ defined by $\omega_{\xi,\mu}(x) = < x\xi, \mu >$ is a bounded linear functional on $B(H)$.

(ii) Prove that the map $\omega_{\xi,\xi} = \omega_\xi$, $\xi \in H$, is positive on $B(H)$ and is a state when $\|\xi\| = 1$.

(iii) If $B(H)_*$ is the norm closure of the vector subspace $B(H)_\sim$ of $B(H)^*$, prove that $B(H) \simeq (B(H)_*)^*$.

Fourth Question

Let A be a C*-algebra. Prove that

(i) a linear functional ρ on A is positive if and only if $\rho(I) = \|\rho\|$.
(ii) if $x \in A$ and $\lambda \in \sigma(x)$, then there is a state ρ of A such that $\rho(x) = \lambda$.

Fifth Question

Let E and F be Banach spaces and let $\phi : E \to F$ be a bounded linear operator.

If $\phi^* : F^* \to E^*$ is the adjoint of ϕ, prove that

(i) ϕ^* is $\sigma(F^*, F) - \sigma(E^*, E)$- continuous and $\|\phi^*\| = \|\phi\|$, (*ii*) if ϕ is an isometry, then ϕ^* maps the closed unit ball $(F^*)_1$ of F^*onto the closed unit ball $(E^*)_1$ of E^*,

(*iii*) if M is a subspace of E, then M° is a $\sigma(E^*, E)$-closed subspace of $E^*$and $(M^\circ)_\circ = \overline{M}^{norm}$.

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