# Continuous Distribution

## Exponential Distribution

Q7: Let X be an **exponential** random variable with parameter $\theta = ln(3)$. Compute the following probability: $P(2 \le X \le 4)$.

**Solution of Q7:**

$$X \sim exp(\ \theta\ ) \Rightarrow \quad f(x) = \theta e^{-\theta x} \quad ; \quad F(x) = 1 - e^{-\theta x} \quad , x > 0$$

$$\boldsymbol{X \sim exp(\ \ \theta = ln(3)\ )}$$

$$\boldsymbol{f(x) = ln(3)\ e^{-ln(3)\ x}} \ ; \ \boldsymbol{x > 0}$$

$$\boldsymbol{F(x) = 1 - e^{-ln(3)\ x}} \quad ; \boldsymbol{x > 0}$$

$$pdf:\ P(2 \le x \le 4) = \int_2^4 f(x)dx = \ln(3) \int_2^4 \mathrm{e}^{-\ln(3)x}\ \mathrm{dx} = \frac{8}{81} = 0.0988$$

$$cdf:\ P(2 \le x \le 4) = p(X \le 4) - P(X \le 2) = 1 - e^{-(4)\ ln(3)} - (1 - e^{-(2)\ ln(3)}) = 0.0988$$

Q8: Suppose the random variable has an **exponential** distribution with parameter $\theta = 1$. compute $P(X > 2)$.

**Solution of Q8:**

$$X \sim exp(\ \theta\ ) \Rightarrow \quad f(x) = \theta e^{-\theta x} \quad ; \quad F(x) = 1 - e^{-\theta x} \quad , x > 0$$

$$\boldsymbol{x \sim \exp(\theta = 1)}$$

$$f(x) = \boldsymbol{e^{-x}} \ ; \ x > 0$$

$$F(x) = \boldsymbol{1 - e^{-x}} \quad ; x > 0$$

$$pdf:\ P(x > 2) = \int_2^\infty f(x)dx = \int_2^\infty e^{-x}\ dx = -e^{-x}\ \Big|_2^\infty = -(e^{-\infty} - e^{-2}) = e^{-2} = 0.1353$$

$$cdf:\ P(x > 2) = 1 - p(x \le 2) = 1 - (1 - e^{-2}) = e^{-2}$$

Q9: What is the probability that a random variable X is less than its expected value, if X has an **exponential** distribution with parameter $\theta$?

**Solution of Q9:**

$X \sim \exp(\theta)$

$$f(x) = \theta e^{-\theta x} \quad , x > 0 \quad ; \quad F(x) = 1 - e^{-\theta x} \quad , x > 0$$

$$E(x) = \int_0^\infty \theta\ e^{-\theta x}\ dx = \theta\ \frac{\Gamma(2)}{\theta^2} = 1/\ \theta$$

$$P\big(x < E(X)\big) = P\left(x < \frac{1}{\theta}\right) = F\left(\frac{1}{\theta}\right) = 1 - e^{-\theta\left(\frac{1}{\theta}\right)} = 1 - e^{-1} = 0.6321$$

## Gamma Distribution

Q1: Show that the mean and variance of **Gamma** distribution are given by

a) $\mu = \frac{\alpha}{\lambda}$

b) $\sigma^2 = \frac{\alpha}{\lambda^2}$

Solution of Q4: Same example in the book page 269

Q2:Let X be a Gamma random variable with $\alpha = 4$ and $\lambda = \frac{1}{2}$. Compute $P(2 < X < 4)$ ?

**Solution:** Gamma dis : $f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \quad , x \geq 0$

$$P(2 < X < 4) = \int_2^4 \frac{\frac{1}{2} e^{-\frac{x}{2}} \left(\frac{x}{2}\right)^{4-1}}{\Gamma(4)} dx = \frac{1}{2^4 \Gamma(4)} \int_2^4 x^3 e^{-\frac{x}{2}} \, dx$$

$$\therefore \Gamma(4) = 3! = 6$$

By use calculate we get $\frac{1}{96} \int_2^4 x^3 e^{-\frac{x}{2}} \, dx = \mathbf{0.1239}$

Q3:If X has a probability density function given by

$$f(x) = \begin{cases} 4x^2 e^{-2x} & ; x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and the variance?

**Solution :** Gamma dis : $f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \quad , x \geq 0$

$X \sim Gamma\,(\alpha = 3 , \lambda = 2) \Rightarrow \quad E(X) = \frac{\alpha}{\lambda} = \frac{3}{2} \quad ; \quad V(X) = \frac{\alpha}{\lambda^2} = \frac{3}{4}$

**Or** $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx \Rightarrow \quad E(X) = \int_0^{\infty} 4x^3 e^{-2x} \, dx = 4 \frac{\Gamma(4)}{2^4} = \frac{4\,(3!)}{2^4} = \frac{3}{2} = 1.5$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx \Rightarrow E(X^2) = \int_0^{\infty} 4x^4 e^{-2x} \, dx = 4 \frac{\Gamma(5)}{2^5} = \frac{4(4!)}{2^5} = 3$$

$$V(X) = E(X^2) - [E(X)]^2 = 3 - (1.5)^2 = \frac{3}{4} = 0.75$$

Q4:Let X be a gamma random variable with $\alpha = 2$ and $\lambda = 3$. Compute $P(X > 3)$?

**Solution:** Gamma dis : $f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \quad , x \geq 0$

$$f(x) = \frac{3}{\Gamma(2)} (3x)^{2-1} e^{-3x} = \frac{\mathbf{3^2}}{\mathbf{\Gamma(2)}} \mathbf{x\, e^{-3x}}$$

$$P(X > 3) = 1 - P(X < 3) = 1 - \left[\frac{3^2}{\Gamma(2)} \int_0^3 x\, e^{-3x}\, dx\right] = 1 - 0.9988 = \mathbf{0.001234}$$

Q5:Suppose the continuous random variable X has the following pdf:

$$f(x) = \begin{cases} \frac{1}{16} x^2 e^{-\frac{x}{2}} & if \quad x > 0 \\ 0 & otherwise \end{cases}$$

Find $E(X^3)$ ?

**Solution :**

$$E(X^3) = \frac{1}{16} \int_0^\infty x^5 e^{-\frac{x}{2}}\, dx = \frac{1}{16} \frac{\Gamma(6)}{\left(\frac{1}{2}\right)^6} = 480$$

$$E(X^3) = 480$$