Chapter (1): Poisson Process

(1) Exponential distribution:

(1.1)

* We say that X has an exponential distribution with parameter $\lambda > 0$ if:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; \; x \geq 0 . \\ 0 & ; \; x < 0 . \end{cases}$$

* $$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & ; \; x \geq 0 \\ 0 & ; \; x < 0 . \end{cases}$$

* moment generating function:

$$\varphi_X(t) = E\, e^{tX} = \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, dx .$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} \, dx = \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_0^\infty$$

$\boxed{t < \lambda}$

$$= \frac{\lambda}{\lambda - t} .$$

* Mean: $E(X) = \varphi_X'(0)$

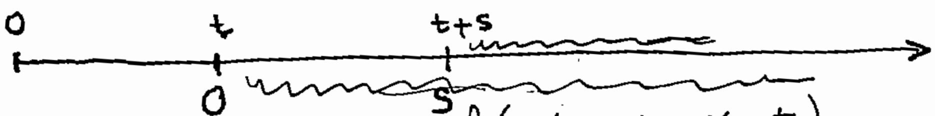
$$\varphi_X'(t) = \frac{\lambda}{(\lambda-t)^2} \; ; \quad \varphi_X''(t) = \frac{2\lambda}{(\lambda-t)^3}$$

$$E(X) = \frac{1}{\lambda} \; ; \quad E(X^2) = \frac{2}{\lambda^2} .$$

* Variance: $\operatorname{var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} .$

(1.2) Memoryless Property:

$$P(X > s+t \mid X > t) = P(X > s) .$$



$$P(X > s+t \mid X > t) = \frac{P(X > s+t \, ; \, X > t)}{P(X > t)}$$

$$= \frac{P(X > s+t)}{P(X > t)} = \frac{1 - F_X(s+t)}{1 - F_X(t)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = 1 - F_X(s)$$

$$= P(X > s) .$$

$* \quad P(X > s+t) = P(X > t) \cdot P(X > s)$.

Example: Suppose the amount of time ~~spend~~ spend in a bank by a costumer is exponentially distributed with mean 10 mns.

1- Find the probability that a costumer will spend more than 15 mns ?

2- ________, given that he is still in the bank after 10 mns ?

$$X \sim Exp(\lambda), \quad \lambda = \frac{1}{10} = 0.1$$

1- $P(X > 15) = 1 - F(15) = e^{-0.1 * 15} = e^{-1.5}$

2- $P(X > 15 \mid X > 10) = P(X > 5) = e^{-0.1 * 5} = e^{-0.5}$

1.3 Compounded exponential distribution:

let $X^1, X^2, \dots, X^n$ independent and exponentially distributed with parameter λ.

let define $X = X^1 + \dots + X^n$.

Then X has a Gamma distribution with parameters n and λ:

$$f_X(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} \cdot e^{-\lambda x}; \quad x \geq 0.$$

$$\varphi_X(t) = E e^{t(X^1 + \dots + X^n)} = E\left(e^{tX^1} \cdots e^{tX^n}\right)$$

$$= E e^{tX^1} \cdots E e^{tX^n} = \left(\frac{\lambda}{\lambda - t}\right)^n, \quad \underline{\underline{t < \lambda}}.$$

let $Y \sim Gamma(\lambda, n)$.

$$\varphi_Y(t) = \int_0^\infty e^{tx} \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} dx.$$

$$= \frac{\lambda^n}{(n-1)!} \int_0^\infty x^{n-1} e^{-(\lambda - t)x} dx.$$

$$= \frac{\lambda^n}{(\lambda - t)^n} = \left(\frac{\lambda}{\lambda - t}\right)^n.$$

* let $X_1^* \sim Exp(\lambda_1^*)$; $X_2^* \sim Exp(\lambda_2^*)$; X_1 and X^2 are independent.

$$P(X_1^* < X_2^*) = \iint_{x_1^* < x_2} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} dx_1 dx_2$$

$$= \int_0^\infty dx_2 \lambda_2 e^{-\lambda_2 x_2} \left(\int_0^{x_2} \lambda_1 e^{-\lambda x_1} dx_1 \right)$$

$$= \int_0^\infty dx_2 \lambda_2 e^{-\lambda_2 x_2} (1 - e^{-\lambda_1 x_2}) = \int_0^\infty dx_2 \left[\lambda_2 e^{-\lambda_2 x_2} - \lambda_2 e^{-\lambda_2 x_2} e^{-\lambda_1 x_2} \right]$$

$$= 1 - \lambda_2 \int_0^\infty e^{-(\lambda_1+\lambda_2) x_2} dx_2 .$$

$$= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2} .$$

if $\lambda_1 = \lambda_2 = \lambda$: $P(X_1 < X_2) = \frac{1}{2}$.

* Convolution of exponential distributions :

let f and g two real functions.
we define the convolution of f and g by :

$$f*g(x) = \int f(x-y)\, g(y)\, dy .$$

(1) $X_1 \sim Exp(\lambda_1)$; $X_2 \sim Exp(\lambda_2)$, X_1 and X_2 are independent.

$X = X_1 + X_2$.
what is the density function of X ?

$f_X = ?$

$$F_X(x) = P(X \le x) = P(X_1 + X_2 \le x)$$

$$= \int P(X_1 + X_2 \le x \mid X_1 = y) f_{X_1}(y)\, dy .$$

$$= \int P(X_2 \le x - y) f_{X_1}(y)\, dy .$$

$$= \int F_{X_2}(x-y) f_{X_1}(y)\, dy .$$

$$f_X(x) = F_X'(x) = \int F_{X_2}'(x-y) f_{X_1}(y)\, dy = \int f_{X_2}(x-y) f_{X_1}(y)\, dy$$

$$= (f_{X_1} * f_{X_2})(x) .$$

$$\boxed{E(X) = \int E(X \mid Y=y)\, f_Y(y)\, dy .}$$

$X = X_1 + X_2$, X_1 and X_2 are independent.

$$\boxed{f_X = f_{X_1} * f_{X_2}}$$

$$f_{X_1}(x) = \lambda_1 e^{-\lambda_1 x}, \; x \geq 0, \quad f_{X_2}(x) = \lambda_2 e^{-\lambda_2 x}; \; x \geq 0.$$

$$f_X(x) = \int_0^x \lambda_1 e^{-\lambda_1 y} \lambda_2 e^{-\lambda_2 (x-y)} \, dy. \qquad x - y \geq 0, \; y \leq x.$$

$\boxed{\lambda_1 \neq \lambda_2}$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 x} \int_0^x e^{-(\lambda_1 - \lambda_2) y} \, dy$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 x} \frac{-1}{\lambda_1 - \lambda_2} e^{-(\lambda_1 - \lambda_2) y} \Big|_0^x$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 x} \frac{-1}{\lambda_1 - \lambda_2} \left(e^{-(\lambda_1 - \lambda_2) x} - 1 \right).$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 x} - e^{-\lambda_2 x} \right) \qquad \underline{x \geq 0.}$$

X is called hypoexponential random variable.

* $X = X_1 + X_2 + \dots + X_n$: $\lambda_i \neq \lambda_j$, $i, j = 1 \dots n$.

$$f_X(x) = \sum_{i=1}^{n} C_{i,n} \underline{\lambda_i e^{-\lambda_i x}} \; ; \; x \geq 0.$$

$$C_{i,n} = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

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(2) Poisson Process:

X_t = continuous time stochastic process if t takes its value in an interval and X_t is a random variable.

(2.1) Counting process:

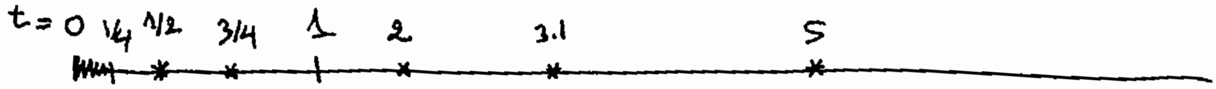
A counting process $(X_t)_{t \geq 0}$ is a process counting the number of times, a certain "event" has occured before time t.

X_t is a discrete random variable.

$X_{0.5}$, $\underline{X_{2/3}}$

Examples:

① Let $X_t =$ be the number of Costumers arriving at a store before time t.



$X_{1/4}$ = number of Costumers arriving at the first quarter.

$X_{1/4} = 0$.

$X_1 =$ ______________ at the 1st hour.

$= 2$.

$X_4 = 4$

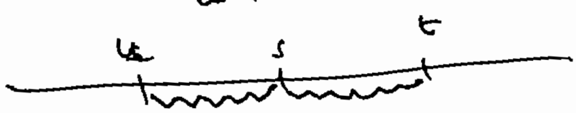
② X_t = number of times, a printer is out of service before time t.

2.2 Poisson Process:

A Poisson Process $(N_t)_{t\geq 0}$ is a couting process such that:

(i) $N_0 = 0$.

(ii) the process has undependent increments: $N_t - N_s$, $N_s - N_u$ are independent. $u < s < t$.



(iii) $N_t \sim Poi(\lambda t)$.

λ is called the rate of the process N.

$$P(N_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \; ; \; E(N_t) = \lambda t$$

$$Var(N_t) = \lambda t.$$

Exercise: let the Poisson Process (N_t) with rate λ. Compute $E(N_t N_s)$, $s<t$?

$$E(N_t N_s) = E\big(\underbrace{(N_t - N_s) + N_s}\big) N_s$$

$$= E(N_t - N_s)(N_s - N_0) + E(N_s)^2$$

$$= E(N_t - N_s)E(N_s) + (\lambda s)^2 + \lambda s$$

$$\boxed{E(X^2) = (E(X))^2 + Var(X)}$$

$N_t - N_s$, $N_s - N_0$ are independent.

$$= \lambda(t-\cancel{s})\lambda s + \cancel{(\lambda s)^2} + \lambda s$$

$$= \lambda^2 st + \lambda s = \lambda s(1 + \lambda t)$$

$$N_t - N_s \overset{"}{\sim} N_{t-s}$$

Example: Suppose the number of costumers arriving at a store, is a Poisson Process with rate $\lambda = 4$ per hour.

a) Find the Probability that only 2 costumers arrive at the first $\frac{1}{2}$ hour.

b) Find the Probability that 5 costumers arrive before 2 hours given that only one costumer arrive before 1 hour?

a) $$P(N_{1/2} = 2) = \frac{e^{-\frac{1}{2}\times 4}\left(\frac{1}{2}\times 4\right)^2}{2!} = 2e^{-2}$$

b) $$P(N_2 = 5 \mid N_1 = 1) = \frac{P(N_2 = 5;\, N_1 = 1)}{P(N_1 = 1)}$$

$$P(N_1=1) = e^{-4}\frac{4^1}{1!} = 4e^{-4}.$$

$$P(N_2=5;\, N_1=1) = P\big((N_2-N_1)+N_1=5;\ N_1=1\big)$$

$$= P(N_2-N_1=4;\ N_1=1)$$

N_1

$$= P(N_2-N_1=4)\,P(N_1=1).$$

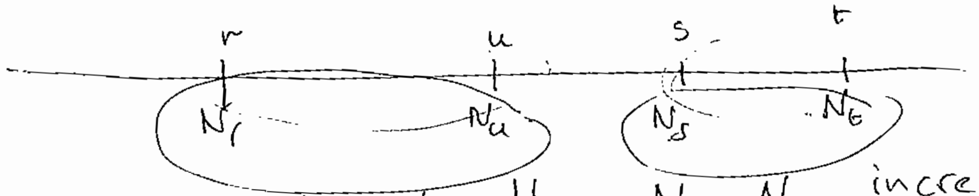
$$= P(N_1=4)\ 4e^{-4}$$

$$= e^{-4}\frac{4^4}{4!}\ 4e^{-4}.$$

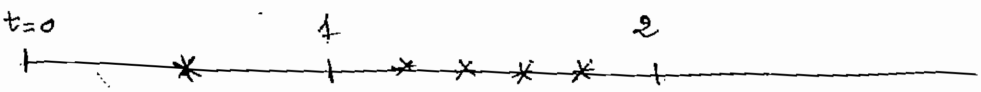
$$P(N_2=5\,|\,N_1=1) = e^{-4}\frac{4^4}{4!}$$

$$N_t - N_s \overset{\neq}{\sim} N_{t-s}$$

- $N_t - N_s$, $N_s - N_0$ are independent



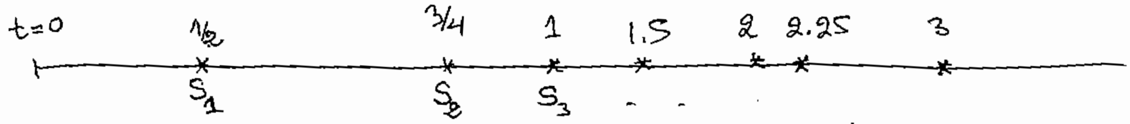
$N_u - N_r \perp\!\!\!\perp N_t - N_s$ <u>increments</u>



$$P(N_2=5\,|\,N_1=1) = P(N_2-N_1=4) = P(N_1=4) = e^{-4}\frac{4^4}{4!}$$

(2.3) Arrival times and Interarrival times:

* Arrival times:



S_1 = time at which the "event" occured for the first time.

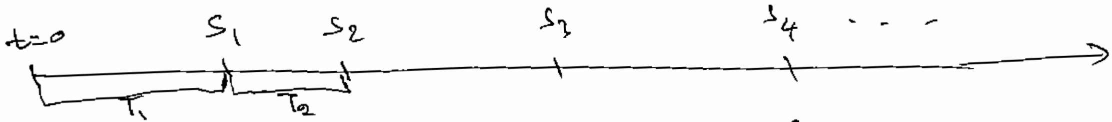
⋮

S_n = time at which the "event" occured for the n^{th} time.

Distribution of S_n = ?

Inter-arrival time:

*



$T_1 = S_1$, $T_2 = S_2 - S_1$, ..., $T_n = S_n - S_{n-1}$

* Distribution of T_1:

$$P(T_1 \le t) = P(S_1 \le t) = P(N_t \ge 1).$$

$$= 1 - P(N_t = 0) = 1 - e^{-\lambda t}\frac{(\lambda t)^0}{0!}$$

$$= 1 - e^{-\lambda t}.$$

$T_1 \sim Exp(\lambda)$.

Distribution of T_2:

*

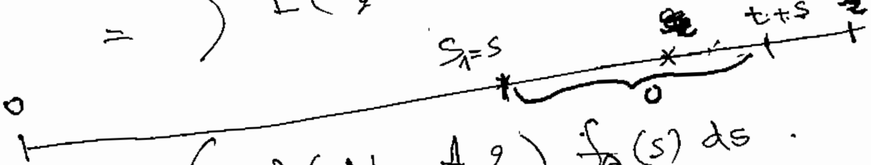
$$P(T_2 \le t) = P(S_2 - S_1 \le t)$$

$$= \int P(S_2 - S_1 \le t \mid S_1 = s)\, f_{S_1}(s)\, ds.$$

T_1, T_2 indep.

S_1, S_2 not indep

$$= \int P(S_2 \le s+t \mid S_1 = s)\, f_{S_1}(s)\, ds.$$



$$= \int P(N_{t+s} \ge 2)\, f_{S_1}(s)\, ds.$$

$$= \int_0^\infty \left(1 - \left(e^{-\lambda(t+s)} + e^{-\lambda(t+s)}\lambda(t+s)\right)\right) \lambda e^{-\lambda s}\, ds$$

$$= \int_0^\infty \left(1 - P(N_{t+s}=0) - P(N_{t+s}=1)\right) \lambda e^{-\lambda s}\, ds.$$

$$= \int_0^\infty \lambda e^{-\lambda s}\, ds - \int_0^\infty e^{-\lambda(t+s)} \lambda e^{-\lambda s}\, ds - \int_0^\infty \lambda(t+s) e^{-\lambda(t+s)} \lambda e^{-\lambda s}\, ds$$

$$* = \int_0^\infty 1\,\lambda e^{-\lambda s}\, ds - \int_0^\infty \dots \quad \lambda^2 t e^{-\lambda t}\int_0^\infty e^{-2\lambda s}\, ds - \lambda^2 e^{-\lambda t}\int_0^\infty s e^{-2\lambda s}\, ds$$

$$= 1 - \lambda e^{-\lambda t}\int_0^\infty e^{-2\lambda s}\, ds - \lambda^2 t e^{-\lambda t}\frac{1}{2\lambda} - \lambda^2 e^{-\lambda t}\frac{1}{2\lambda}$$

$$= 1 - \lambda e^{-\lambda t}\frac{1}{2\lambda} - \dots$$

$$= \int_0^\infty \lambda e^{-\lambda s} ds - \int_0^\infty e^{-\lambda(t+s)} \lambda e^{-\lambda s} ds - \int_0^\infty \lambda t\, e^{-\lambda(t+s)} \lambda e^{-\lambda s} ds - \int_0^\infty \lambda s\, e^{-\lambda(t+s)} \lambda e^{-\lambda s} ds$$

$$= 1 - \lambda e^{-\lambda t} \int_0^\infty e^{-2\lambda s} ds - \lambda^2 t e^{-\lambda t} \int_0^\infty e^{-2\lambda s} ds - \lambda^2 e^{-\lambda t} \int_0^\infty s\, e^{-2\lambda s} ds$$

$$= 1 - \lambda e^{-\lambda t} \frac{1}{2\lambda} - \lambda^2 t e^{-\lambda t} \frac{1}{2\lambda} - \lambda^2 e^{-\lambda t} \frac{1}{2\lambda} \frac{1}{2\lambda}$$

$$= 1 - e^{-\lambda t} \left(\frac{1}{2} + \frac{\lambda t}{2} + \frac{1}{4} \right).$$

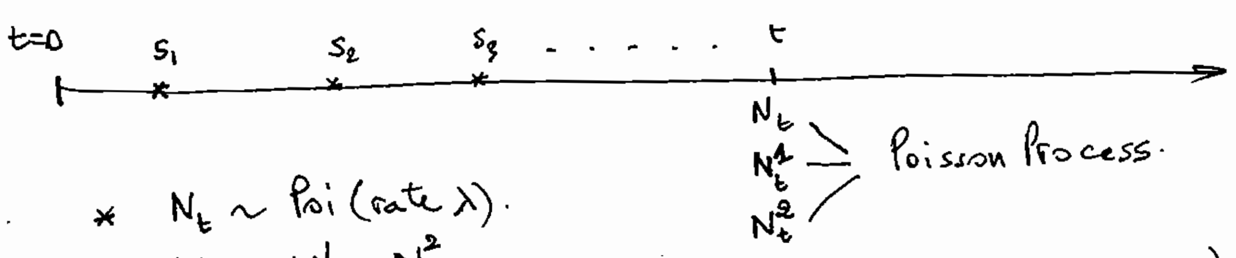
$T_2 \sim Exp(\lambda)$.

... $T_n \sim exp(\lambda)$.

Theorem: * each T_n has an exponential distribution with parameter λ.

** $T_1, T_2, \ldots$ are independent.

theorem: $S_n = T_1 + \cdots + T_n \sim Gamma(\lambda, n-1)$.



* $N_t \sim Poi(\text{rate } \lambda)$.
* $N = N^1 + N^2$
* $N^1 \sim Poi(\lambda p)$
* $N^2 \sim Poi(\lambda(1-p))$.

$p = \mathbb{P}(\text{Customer is a man})$.