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Family of Optimal Eighth-Order of Convergence for Solving Nonlinear Equations

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Abstract

In this paper, a new family of optimal eighth-order iterative methods are presented. The new family is developed by combining Traub-Ostrowski's fourth-order method adding Newton's method as a third step and using the forward divided difference and three real-valued functions in the third step to reduce the number of function evaluations. We employed several numerical comparisons to demonstrate the performance of the proposed method.

Keywords Convergence order; Efficiency index; Iterative methods; Nonlinear equations; Optimal eighth-order.

1. Introduction

Solving of nonlinear equations is one of the oldest and most important problems in numerical analysis. In scientific departments, a need arises to solve nonlinear equations. Newton's method is an important and basic method [9] for identifying a simple root of a nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f : D \subset R \to R$ for an open interval D. The classical Newton method is given as (NM)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

which converges quadratically [9]. In recent years, many researchers worked to develop several iterative methods for solving nonlinear equations. For example, the method of weight functions in iterative methods for a simple root has been presented [3, 4, 10, 13]. Recently, there several eighth-order methods have been proposed in [5, 6]. Optimal three-step methods with eighth-order convergence developed in [1].

In this paper, we present a new family method that uses Traub-Ostrowski's method in the first two steps, given by, (TOM)

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(y_n)-f(x_n)}{2f(y_n)-f(x_n)} \frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{3}$$

which is of fourth-order of convergence [15].

Theorem 1.

Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_s(x)$be iterative functions with the orders $p_1, p_2, \ldots, p_s$, respectively. Then the composition of iterative functions $\varphi_1\left(\varphi_2\left(\ldots\left(\varphi_s(x)\right)\ldots\right)\right)$, defines the iterative method of the order $p_1 p_2 \cdots p_s$[11].

By using theorem 1, we add Newton's method as a third step as follows, (TONM)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \tag{4}$$

the efficiency index (EI) is defined by $E = p^{1/n}$, where p is the order of convergence and n is the number of total function and derivative evaluations per iteration [14]. According to the optimality, the optimal order of any multipoint iterative method is given by 2^{n-1} [7]. Thus, the efficiency index of the optimal fourth-order (TOM), i.e. method (3) is $4^{1/3} \approx 1.5874$ and for (TONM), i.e. method (4) is $8^{1/5} \approx 1.5157$. Method (4) is not optimal and requires five evaluations. Moreover, method (4) has an order of eight. The aim of this paper is to reduce the number of function evaluations of method (4) to four to make method (4) an optimal eighth-order of convergence by replacing $f'(z_n)$ with $f[y_n, z_n]$ where forward divided difference, $f[y_n, z_n] = \frac{f(z_n) - f(y_n)}{z_n - y_n}$ and the equivalent construction of weighted functions.

In section 2, we present a new family of optimal eighth-order methods. In section 3, numerical comparisons are made to illustrate the efficiency and performance of the newly proposed method. Finally, the conclusion of the paper is presented.

2. Method and Convergence Analysis

The order of convergence of the proposed method (4) is eight which is clearly not optimal. To construct an optimal eighth-order method without using more evaluations, we present a new family of the optimal eighth-order as follows, (ASM)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \{A(t_1) + B(t_2) + C(t_3)\} \frac{f(z_n)}{f[y_n, z_n]}, \tag{5}$$

where $A(t_1), B(t_2), C(t_3)$ are three real-valued weight functions, and

$$t_1 = \frac{f(y_n)}{f(x_n)}, t_2 = \frac{f(z_n)}{f(y_n)}, t_3 = \frac{f(z_n)}{f(x_n)}. \tag{6}$$

The weight functions A, B and C should be chosen such that the order of convergence of method (5) arrives at an optimal level of eight. In the following theorem we prove that method (5) has an optimal eighth-order of convergence under conditions for the weighted functions that improve the method (5) to an optimal of order eight.

Theorem 2.

Let the function $f: D \subseteq \mathbb{R} \to \mathbb{R}$ have a simple root on the open interval *D*. If initial point x_0 is sufficiently close to α, then the method described by (5) has an optimal eighth-order of convergence and converges under the following conditions

$A(0) = 1, A'(0) = 0, A''(0) = 2, A^{(3)}(0) = 12;\ B(0) = -1, B'(0) = B''(0) = B^{(3)}(0) = 0;\ C(0) = 1,$
$C'(0) = 2, C''(0) = C^{(3)}(0) = 0.$

Proof:

Let $e_n = x_n - \alpha$ be the error at the nth iteration. By Taylor expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9)], \tag{7}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!\, f'(\alpha)}, k = 2,3, \dots$.

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + 9c_9 e_n^8 + O(e_n^9)]. \tag{8}$$

Dividing (7) by (8), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + \ldots + (19c_2c_7 - 7c_8 - 118c_5c_2c_3 + 348c_4c_3c_2^2 - 64c_2^7 - 64c_2c_4^2 - 176c_4c_2^4 + 92c_5c_2^3 + 27c_6c_3 - 44c_6c_2^2 + 304c_3c_2^5 - 75c_4c_3^2 + 31c_5c_4 + 135c_2c_3^3 - 408c_3^2c_2^3)e_n^8 + O(e_n^9) \tag{9}$$

Now, from (9), we have

$$y_n = \alpha + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + \ldots + (7c_8 - 19c_2c_7 + 118c_5c_2c_3 - 348c_4c_3c_2^2 + 64c_2^7 + 64c_2c_4^2 + 176c_4c_2^4 - 92c_5c_2^3 - 27c_6c_3 + 44c_6c_2^2 - 304c_3c_2^5 + 75c_4c_3^2 - 31c_5c_4 - 135c_2c_3^3 + 408c_3^2c_2^3)e_n^8 + O(e_n^9). \tag{10}$$

From (10), we obtain

$$f(y_n) = f'(\alpha)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + \ldots + (7c_8 - 19c_2c_7 + 134c_5c_2c_3 - 455c_4c_3c_2^2 + 144c_2^7 + 73c_2c_4^2 + 297c_4c_2^4 - 134c_5c_2^3 - 27c_6c_3 + 54c_6c_2^2 - 552c_3c_2^5 + 75c_4c_3^2 - 31c_5c_4 - 147c_2c_3^3 + 582c_3^2c_2^3)e_n^8] + O(e_n^9). \tag{11}$$

In view of (6), (8), (9) and (11), we obtain

$$\frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}\frac{f(x_n)}{f'(x_n)} = e_n + (c_2c_3 - c_2^3)e_n^4 + \ldots + (5c_2 - 68c_5c_2c_3 + 209c_4c_3c_2^2 - 36c_2^7 - 37c_2c_4^2 - 101c_4c_2^4 + 51c_5c_2^3 + 13c_6c_3 - 20c_6c_2^2 + 178c_3c_2^5 - 50c_4c_3^2 + 17c_5c_4 + 91c_2c_3^3 - 252c_3^2c_2^3)e_n^8 + O(e_n^9). \tag{12}$$

Combining (9), (10), (11) and (12), we have

$$z_n = \alpha + (-c_2c_3 + c_2^3)e_n^4 + \ldots + (-5c_2 + 68c_5c_2c_3 - 209c_4c_3c_2^2 + 36c_2^7 + 37c_2c_4^2 + 101c_4c_2^4 - 51c_5c_2^3 - 13c_6c_3 + 20c_6c_2^2 - 178c_3c_2^5 + 50c_4c_3^2 - 17c_5c_4 - 91c_2c_3^3 + 252c_3^2c_2^3)e_n^8 + O(e_n^9). \tag{13}$$

From (13), we get

$$f(z_n) = f'(\alpha)\,[(-c_2c_3 + c_2^3)e_n^4 + \ldots + (-5c_2 + 68c_5c_2c_3 - 209c_4c_3c_2^2 + 37c_2^7 + 37c_2c_4^2 + 101c_4c_2^4 - 51c_5c_2^3 - 13c_6c_3 + 20c_6c_2^2 + 180c_3c_2^5 + 50c_4c_3^2 - 17c_5c_4 - 91c_2c_3^3 + 253c_3^2c_2^3)e_n^8] + O(e_n^9). \tag{14}$$

From (7), (11) and (14), it can be easily determine that

$$f[y_n, z_n] = f'(\alpha) + f'(\alpha)c_2^2e_n^2 + \ldots + O(e_n^9), \tag{15}$$

$$\frac{f(z_n)}{f(y_n)} = (-c_3 + c_2^2)e_n^2 + (4c_2c_3 - 2c_2^3 - 2c_4)e_n^3 + \ldots + O(e_n^9), \tag{16}$$

$$\frac{f(z_n)}{f(x_n)} = (-c_2c_3 + c_2^3)e_n^3 + (9c_3c_2^2 - 5c_2^4 - 2c_2c_4 - 2c_3^2)e_n^4 + \ldots + O(e_n^9). \tag{17}$$

Finally, using (16), (17), (13), (14), (15) and

$A(0) = 1, A'(0) = 0, A''(0) = 2, A^{(3)}(0) = 12;\ B(0) = -1, B'(0) = B''(0) = B^{(3)}(0) = 0;\ C(0) = 1,$

$C'(0) = 2, C''(0) = C^{(3)}(0) = 0,$ we obtain the error expression

$$e_{n+1} = \alpha + (-c_4c_3c_2^2 + 7c_2^7 + c_4c_2^4 + 4c_3^2c_2^3 - 11c_3c_2^5)e_n^8 + O(e_n^9). \tag{18}$$

The theorem is proved.

Particular case. Let

$$A(t_1) = 1 + t_1^2 + 2t_1^3 + \alpha t_1^4, \tag{19}$$

$$B(t_2) = -1 + \beta t_2, \tag{20}$$

$$C(t_3) = 1 + 2t_3 + \gamma t_3^2, \tag{21}$$

where $\alpha, \beta, \gamma \in R$; then the method becomes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \{\{1 + t_1^2 + 2t_1^3 + \alpha t_1^4\} + \{-1 + \beta t_2\} + \{1 + 2t_3 + \gamma t_3^2\}\} \frac{f(z_n)}{f[y_n, z_n]}, \tag{22}$$

where $t_1 = \frac{f(y_n)}{f(x_n)}$, $t_2 = \frac{f(z_n)}{f(y_n)}$, and $t_3 = \frac{f(z_n)}{f(x_n)}$.

3. Numerical Results

In this section, we present several numerical tests to illustrate the efficiency of the new method. We compared the performance of two cases of the new optimal eighth-order method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \{\{1 + t_1^2 + 2t_1^3 + \alpha t_1^4\} + \{-1 + \beta t_2\} + \{1 + 2t_3 + \gamma t_3^2\}\} \frac{f(z_n)}{f[y_n, z_n]}, \tag{23}$$

where $\alpha, \beta,$ and $\gamma = 0$, (ASM1) and where $\alpha = 1, \beta = 0$, and $\gamma = -2$, (ASM2), with Newton's method (NM), method (2), Traub-Ostrowski's method (TOM), method (3), and some optimal eighth-order methods, as well as the method (BWRM) proposed by Bi–Wu–Ren in [2], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(x_n) + (2+\gamma) f(z_n)}{f(x_n) + \gamma f(z_n)} \frac{f(z_n)}{f[z_n, y_n] + (z_n - y_n) f[z_n, x_n, x_n]}, \tag{24}$$

where $\gamma = 1$, the method (LWM) proposed by Liu and Wang in [8], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left[\left(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} \right)^2 + \frac{f(z_n)}{f(y_n) - \mu f(z_n)} + \frac{4f(z_n)}{f(x_n) + \beta f(z_n)} \right], \tag{25}$$

where $\beta = \mu = 1$, and the method (SM) proposed by Sharma in [12], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \left[1 + \frac{f(z_n)}{f(x_n)} + \left(\frac{f(z_n)}{f(x_n)} \right)^2 \right] \frac{f[x_n, y_n] f(z_n)}{f[x_n, z_n] f[y_n, z_n]}, \tag{26}$$

The test functions and their exact root α are displayed with only nine decimal digits as follows

$f_1(x) = \log(x) + \sqrt{x} - 5,\ \alpha_1 = 8.3094326942,$

$f_2(x) = x^3 + 4x^2 - 15,\quad \alpha_2 = 1.6319808055,$

$f_3(x) = \sin x - \frac{x}{2},\ \alpha_3 = 1.8954942670,$

$f_4(x) = x^3 - \sin^2(x) + 3\cos x + 5,\ \alpha_4 = -1.5826870457,$

$f_5(x) = \sin x - \frac{x}{3},\ \alpha_5 = 2.2788626600.$

All computations were performed with MATLAB (R2013a) using 2000 digits, floating point (i.e. digits:= 2000). The stopping criteria were

i. $|x_{n+1} - x_n| \leq 10^{-15}$,
ii. $|f(x_{n+1})| \leq 10^{-15}$.

Displayed In the Table1, the number of iterations are denoted by (IT), the number of function evaluations denoted by (NFE) and the values of $|f(x_{n+1})|$ and $|x_{n+1} - x_n|$. are computed. Moreover, the computational order of convergence (COC) approximated as in [15], is also displayed in Table1, defined as

$$\rho = \frac{ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{ln|(x_n - \alpha)/(x_{n+1} - \alpha)|}.$$

Table 1. Comparison of various iterative methods.

	NM	TOM	LWM	SM	BWRM	ASM1	ASM2
$f_1(x), x_0 = 11.9$							
IT	6.0	4.0	3.0	3.0	3.0	3.0	**3.0**
COC	2.0	4.0	7.919	8.0	7.897	8.0	**7.929**
NFE	12	12	12	12	12	12	**12**
$\|f(x_{n+1})\|$	1.397e-54	1.876e-237	2.985e-398	3.924e-492	9.443e-404	6.922e-414	**2.387e-426**
$\|x_{n+1} - x_n\|$	1.059e-26	1.048e-58	3.119e-49	8.202e-61	6.327e-50	3.737e-51	**1.077e-52**
$f_2(x), x_0 = 2$							
IT	6.0	4.0	3.0	3.0	3.0	3.0	**3.0**
COC	2.0	4.0	7.907	8.0	7.875	8.0	**7.910**
NFE	12	12	12	12	12	12	**12**
$\|f(x_{n+1})\|$	8.230e-54	1.025e-228	3.957e-386	9.607e-427	7.328e-467	1.501e-400	**4.706e-404**
$\|x_{n+1} - x_n\|$	9.618e-28	9.681e-5	7.706e-49	7.772e-54	7.913e-59	1.270e-50	**4.742e-51**
$f_3(x), x_0 = 2$							
IT	5.0	4.0	3.0	3.0	3.0	3.0	**3.0**
COC	2.0	4.0	7.961	8.0	7.980	8.0	**7.963**
NFE	10	12	12	12	12	12	**12**
$\|f(x_{n+1})\|$	1.478e-40	1.050e-78	4.625e-553	2.490e-594	4.530e-652	9.383e-575	**4.097e-578**
$\|x_{n+1} - x_n\|$	1.766e-20	4.852e-20	1.054e-69	8.714e-75	6.803e-82	2.218e-72	**8.587e-73**
$f_4(x), x_0 = -1$							
IT	6.0	4.0	3.0	3.0	3.0	3.0	**3.0**
COC	2.0	4.0	8.361	8.0	8.275	8.0	**8.331**
NFE	12	12	12	12	12	12	**12**
$\|f(x_{n+1})\|$	7.021e-38	3.333e-165	1.930e-230	3.019e-285	1.250e-278	3.282e-230	**1.431e-244**
$\|x_{n+1} - x_n\|$	1.371e-19	1.03697e-41	2.932e-29	4.845e-36	3.294e-35	3.22043e-29	**5.278e-31**
$f_5(x), x_0 = 2$							
IT	6.0	4.0	3.0	3.0	3.0	3.0	**3.0**
COC	2.0	4.0	8.166	8.0	8.070	8.0	**8.163**
NFE	12	12	12	12	12	12	**12**
$\|f(x_{n+1})\|$	4.204e-58	7.182e-210	4.682e-334	9.833e-386	1.914e-454	1.362e-352	**2.110e-360**
$\|x_{n+1} - x_n\|$	1.052e-28	9.241e-53	3.168e-42	1.303e-48	4.211e-57	1.689e-44	**1.809e-45**

4. Conclusions

In this paper, a new optimal eighth-order iterative family of methods for solving nonlinear equations was developed. The new proposed family is obtained by replacing $f'(z_n)$ with $f[y_n, z_n]$ and the equivalent construction of weighted functions to reduce the number of function evaluations of (TONM), i.e. method (4), to four. Numerical results are given to illustrate the efficiency and performance of the newly proposed method.

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