King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2-1444H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 3)

Q1: [3+4]

Consider the model of the total dollars paid on a medical malpractice policy in one year that is defined by an insurance company as

$$F(x)=\begin{cases}0, & x<0,\\ 1-0.3e^{-0.00001x}, & x\geq 0.\end{cases}$$

(i) Determine the survival, density, and hazard rate functions.

(ii) Determine the mean excess loss and limited expected value functions.

Q2: [2+4]

(a) Consider a frailty model with frailty random variable Λ, such that $a(x)=\dfrac{1}{x+1}$, $x>0$.

Find the conditional survival function of X.

(b) Let X have a Pareto distribution, where $F(x)=1-\left(\dfrac{\theta}{x+\theta}\right)^{\alpha}$. Determine the cdf of the inverse, transformed, and inverse transformed distributions. Identify the names of these distributions.

Q3: [2+2+1.5+2+2.5]

There are two types of drivers. Good drivers make up 70% of the population and in one year have zero claims with probability 0.6, one claim with probability 0.3, and two claims with probability 0.1. Bad drivers make up the other 30% of the population and have zero, one, or two claims with probabilities 0.4, 0.3, and 0.3, respectively.

(a) Describe this process by using the concept of the risk parameter Θ.

(b) For a particular policyholder, suppose that we have observed $x_1=0$ and $x_2=1$ for past claims.

Determine each of the following:

(i) The posterior distribution of $\Theta \mid X_1=0,\ X_2=1$

(ii) The predictive distribution of $X_3 \mid X_1=0,\ X_2=1$

(iii) The Bayesian premium estimate

(iv) The Bühlmann premium estimate

Q4: [4+4]

(a) The average claim size for a group of insureds is 1500, with a standard deviation of 7500. Assume that claim counts have the Poisson distribution. Determine the expected number of claims so that the total loss will be within 5% of the expected total loss with probability 0.90.

(b) Suppose that the number of claims from m_j policies is N_j in year j for a group policyholder with risk parameter Θ has a Poisson distribution with mean $m_j\Theta$, that is, for $j = 1,...,n$,

$$\Pr(N_j = x|\Theta = \theta) = \frac{(m_j\theta)^x e^{-m_j\theta}}{x!}, \ x = 0,1,2,...,$$

where Θ has a gamma distribution with parameters α and β. Determine the Bühlmann- Straub estimate of the expected number of claims in year $n+1$ for the m_{n+1} policies.

Q5: [6+3]

(a) A sample of 100 losses revealed that 62 were below 1,000 and 38 were above 1,000. An exponential distribution with mean θ is considered. Using only the given information, determine the maximum likelihood estimate of θ. Now suppose you are also given that the 62 losses that were below 1,000 totaled 28,140, while the total for the 38 above 1,000 remains unknown. Using this additional information, determine the maximum likelihood estimate of θ.

(b) Suppose, you have observed the following five claim severities:

16.0, 20.2, 20.0, 19.0 and 36.8. Determine the maximum likelihood estimate of μ for the following model.

$$f(x) = \frac{1}{\sqrt{2\pi x}} \exp[-\tfrac{1}{2x}(x-\mu)^2], \ x, \mu > 0$$

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [6+3]

(i) The survival function is

$$S(x) = -F(x)$$
$$\therefore\ S(x) = 0.3e^{-0.00001x}, \quad x \geq 0$$

The density function is

$$f(x) = F^{'}(x) = -S^{'}(x)$$
$$\therefore\ f(x)\ = 0.000003e^{-0.00001x}, \quad x > 0$$

The distribution of this model is mixed, so we can write the probability density function as follows:

$$f(x) = \begin{cases} 0.7, & x = 0, \\ 0.000003e^{-0.00001x}, & x > 0. \end{cases}$$

The hazard rate function

$$h(x) = \frac{f(x)}{S(x)}$$
$$\therefore\ h(x) = 0.00001,\ x > 0$$

The mean excess loss function

$$e(d) = \frac{\int_d^\infty S(x)\ dx}{S(d)}$$
$$= \frac{\int_d^\infty 0.3e^{-0.00001x}\ dx}{0.3e^{-0.00001d}}$$
$$= -100000\frac{[e^{-0.00001x}]_d^\infty}{e^{-0.00001d}}$$
$$\therefore\ e(d) = 100000 \qquad (1)$$

Which is constant function.

To get the limited expected value function $E(X \wedge u)$

First Method

$$E(X \wedge u) = \int_{-\infty}^{u} x f(x) dx + u[1 - F(u)]$$

$$\Rightarrow \; E(X \wedge u) = \int_{0}^{u} x(0.000003)e^{-0.00001x} dx + u(0.3)e^{-0.00001u}$$

$$I = \int_{0}^{u} x(0.000003)e^{-0.00001x} dx$$

$$= 0.000003\left[\frac{xe^{-0.00001x}}{-0.00001}\Bigg|_{0}^{u} - \int_0^u \frac{e^{-0.00001x}}{-0.00001} dx \right]$$

$$= 0.000003\left[\frac{-ue^{-0.00001u}}{0.00001} - \frac{e^{-0.00001u} - 1}{(0.00001)^2} \right]$$

$$\therefore I \; = -0.3ue^{-0.00001u} - 30000(e^{-0.00001u} - 1) \qquad (2)$$

$$\therefore E(X \wedge u) = 30000\left[1 - e^{-0.00001u}\right]$$

Second Method

By using the following formula

$$E(X \wedge u) = \; E(X) - e(u)S(u)$$

To obtain $E(X)$, let $u \rightarrow \infty$ in (2)

$$\therefore \; E(X) = 30000$$

Also, $e(u)$ is determined before in (1), $e(u) = 100000$

$$\therefore \; E(X \wedge u) = 30000 - 100000(0.3e^{-0.00001u})$$
$$= 30000(1 - e^{-0.00001u})$$

Third Method

Simply, we can use the following formula

$$E(X \wedge u) = -\int_{-\infty}^{0} F(x)dx + \int_{0}^{u} S(x)dx$$

$$\Rightarrow \; E(X \wedge u) = 0 + \int_{0}^{u} 0.3e^{-0.00001x}dx$$

$$= 0.3\frac{e^{-0.00001x}}{-0.00001}\bigg|_{0}^{u}$$

$$=30{,}000[1 - e^{-0.00001u}]$$

Q2: [2+4]

(a)

We first find $A(x)$

$$A(x) = \int_0^x a(t)\, dt$$

$$= \int_0^x \frac{dt}{1+t} = \ln(1+x)$$

Thus,

$$S_{X|\Lambda}(x|\lambda) = e^{-\lambda A(x)}$$

$$= e^{-\lambda \ln(1+x)}$$

$$= \frac{1}{(1+x)^{\lambda}}$$

(b)

For Pareto distribution with parameters α, θ $F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}$

For $\tau > 0,\; F_Y(y) = F_X(y^{\tau})$

$$\therefore F_Y(y) = 1 - \left(\frac{\theta}{y^{\tau}+\theta}\right)^{\alpha}$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{1+(y/\theta^{1/\tau})^{\tau}}\right)^{\alpha}$$

Which is the Burr distribution with three parameters $\alpha, \theta^{1/\tau}, \tau$

For $\tau = -1,\ F_Y(y) = 1 - F_X(y^{-1})$

$$\therefore F_Y(y) = 1 - \left[1 - \left(\frac{\theta}{y^{-1} + \theta}\right)^{\alpha}\right]$$

$$= \left(\frac{\theta}{y^{-1} + \theta}\right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{y}{y + \theta^{-1}}\right)^{\alpha}$$

which is the inverse Pareto distribution with parameters α, θ^{-1}

For negative $\tau,\quad F_Y(y) = 1 - F_X(y^{-\tau})$

$$\therefore F_Y(y) = 1 - \left[1 - \left(\frac{\theta}{\theta + y^{-\tau}}\right)^{\alpha}\right]$$

$$= \left(\frac{\theta}{\theta + y^{-\tau}}\right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + \theta^{-1}}\right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + \left(\theta^{-1/\tau}\right)^{\tau}}\right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{\left(y / \theta^{-1/\tau}\right)^{\tau}}{1 + \left(y / \theta^{-1/\tau}\right)^{\tau}}\right)^{\alpha}$$

which is the inverse Burr distribution with three parameters $\alpha, \theta^{-1/\tau}, \tau$

Q3: [2+2+1.5+2+2.5]

(a)

x	$\Pr(X = x \mid \Theta = G)$	$\Pr(X = x \mid \Theta = B)$	θ	$\Pr(\Theta = \theta)$
0	0.6	0.4	G	0.7
1	0.3	0.3	B	0.3
2	0.1	0.3		

(b)

(i)

For the posterior distribution, the posterior probabilities are given by

$$\pi(G \mid 0,1) = \frac{f(0 \mid G) f(1 \mid G)\pi(G)}{f_X(0,1)}$$

where $f_X(0,1) = \sum_\theta f_{X_1|\Theta}(0 \mid \theta) f_{X_2|\Theta}(1 \mid \theta)\pi(\theta)$

$$\begin{aligned} f_X(0,1) &= 0.6(0.3)(0.7) + 0.4(0.3)(0.3) \\ &= 0.162 \end{aligned}$$

$$\pi(G \mid 0,1) = \frac{0.6(0.3)(0.7)}{0.162} = \frac{7}{9} = 0.7778$$

$$\pi(B \mid 0,1) = \frac{0.4(0.3)(0.3)}{0.162} = \frac{2}{9} = 0.2222$$

(ii)

For the predictive distribution, the predictive probabilities are given by

$$\begin{aligned} f_{X_3|X}(0 \mid 0,1) &= \sum_\theta f(0 \mid \theta)\pi(\theta \mid 0,1) \\ &= f(0 \mid G)\pi(G \mid 0,1) + f(0 \mid B)\pi(B \mid 0,1) \\ &= 0.6(0.7778) + 0.4(0.2222) \\ &= 0.55556, \end{aligned}$$

$$\begin{aligned} f_{X_3|X}(1 \mid 0,1) &= \sum_\theta f(1 \mid \theta)\pi(\theta \mid 0,1) \\ &= f(1 \mid G)\pi(G \mid 0,1) + f(1 \mid B)\pi(B \mid 0,1) \\ &= 0.3(0.7778) + 0.3(0.2222) \\ &= 0.3, \end{aligned}$$

and $f_{X_3|X}(2|0,1) = \sum_{\theta} f(2|\theta)\pi(\theta|0,1)$

$$= f(2|G)\pi(G|0,1) + f(2|B)\pi(B|0,1)$$
$$= 0.1(0.7778) + 0.3(0.2222)$$
$$= 0.14444.$$

(iii)

For Bayesian premium estimate

$$E(X_3|0,1) = \mu(\text{G})\pi(\text{G}|0,1) + \mu(\text{B})\pi(\text{B}|0,1)$$

$$\mu(\text{G}) = 0(0.6)+1(0.3)+2(0.1)$$
$$= 0.5$$

$$\mu(\text{B}) = 0(0.4)+1(0.3)+2(0.3)$$
$$= 0.9$$

$$\therefore E(X_3|0,1) = 0.5(0.7778) + 0.9(0.2222)$$
$$= 0.58888$$
$$\simeq 0.6$$

(iv)

For Bühlmann premium estimate

The collective premium is

$$\mu = E[\mu(\Theta)]$$
$$= \mu(G)\pi(G) + \mu(B)\pi(B)$$
$$= (0.5)(0.7) + (0.9)(0.3)$$
$$\therefore\ \mu = 0.62$$

The variance of hypothetical means is

$$a = \text{var}[\mu(\Theta)]$$
$$= \mu^2(G)\pi(G) + \mu^2(B)\pi(B) - \mu^2$$
$$= (0.5)^2(0.7) + (0.9)^2(0.3) - 0.62^2$$
$$\therefore\ a = 0.0336$$

For the process variance v

$$v = E[v(\Theta)]$$
$$= v(G)\pi(G) + v(B)\pi(B)$$

$$v(G) = \text{var}(X_j | G)$$
$$= 0^2(0.6) + 1^2(0.3) + 2^2(0.1) - 0.5^2$$
$$\therefore v(G) = 0.45$$

$$v(B) = \text{var}(X_j | B)$$
$$= 0^2(0.4) + 1^2(0.3) + 2^2(0.3) - 0.9^2$$
$$\therefore v(B) = 0.69$$

$$v = E[v(\Theta)]$$
$$= 0.45(0.7) + 0.69(0.3)$$
$$\therefore v = 0.522$$

$$k = \frac{v}{a}$$
$$= 15.53571429$$

$$Z = \frac{n}{n+k}$$
$$= \frac{2}{2 + 15.53571429}$$
$$\therefore \text{ Z } = 0.11405$$

The Bühlmann premium is

$$E(X_3 | 0,1) = P_c = Z\overline{X} + (1 - Z)\mu$$
$$= 0.11405(0.5) + (1 - 0.11405)(0.62)$$
$$= 0.6063$$
$$\simeq 0.6$$
$$\text{where } \overline{X} = \frac{0+1}{2} = 0.5, \ \mu = 0.62$$

Clearly, this result is closed to Bayesian premium estimate.

Q4: [4+4]

(a)

at $p = 0.90,\ \Phi(y_p) = (1+p)/2 = 0.95$

$\Rightarrow y_p = 1.645$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.645/0.05)^2 = 1082.41$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1+(\tfrac{\sigma}{\theta})^2]$$

where $\sigma^2 = 7500^2,\ \theta = 1500$

$\therefore$ The expected # of claims $= 1082.41[1+\left(\tfrac{7500}{1500}\right)^2]$

$= 28142.66$

(b)

Let $X_j = N_j / m_j$ be the average of claims per individual in year j.

$\because N_j \mid \Theta$ has a Poisson distribution with mean $m_j\Theta$,

$$E(X_j \mid \Theta) = E\left(\frac{N_j}{m_j} \middle| \Theta\right) = \frac{m_j\Theta}{m_j} = \Theta = \mu(\Theta)$$

and

$$Var(X_j \mid \Theta) = Var\left(\frac{N_j}{m_j} \middle| \Theta = \theta\right)$$

$$= \frac{1}{m_j^2} Var(N_j \mid \Theta) = \frac{m_j\Theta}{m_j^2}$$

$$= \frac{\Theta}{m_j} = \frac{\upsilon(\Theta)}{m_j}$$

$\Rightarrow$

$\mu = E[\mu(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of hypothetical means, where $\Theta \sim gamma(\alpha, \beta)$,

$\upsilon = E[\upsilon(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of process variance and $a = Var(\Theta) = \alpha\beta^2$ is the variance of hypothetical means.

$$\therefore k = \frac{\upsilon}{a} = \frac{1}{\beta},\ Z = \frac{m}{m+k} = \frac{m\beta}{m\beta+1}.$$

So, the Bühlmann-Straub estimate for one policyholder is

$$P_c = \frac{m\beta}{m\beta+1}\overline{X} + \left(1 - \frac{m\beta}{m\beta+1}\right)\mu$$

$$= \frac{m\beta}{m\beta+1}\overline{X} + \frac{1}{m\beta+1}\alpha\beta \text{ where } \overline{X} = \text{m}^{-1}\sum_{j=1}^{n} m_j X_j$$

For year $n+1$, the estimate is $m_{n+1}P_c$.

Q5: [6+3]

(a)

For the first part of the pb, we have

$$L(\theta) = \left[F(1000)\right]^{62}\left[1-F(1000)\right]^{38}$$
$$= \left[1-e^{-1000/\theta}\right]^{62}\left[e^{-1000/\theta}\right]^{38}$$

Let $x = e^{-1000/\theta}$, then

$$L(x) = (1-x)^{62}x^{38}$$
$$\Rightarrow$$
$$l(x) = 62\ln(1-x) + 38\ln x$$
$$\Rightarrow$$
$$l'(x) = \frac{-62}{1-x} + \frac{38}{x}$$

Set $l'(x) = 0$, then $\frac{-62x + 38(1-x)}{x(1-x)} = 0$

$$\therefore\ x = 0.38$$
$$\Rightarrow$$
$$0.38 = e^{-1000/\theta}$$
$$\therefore\ \hat{\theta} = -1000/\ln 0.38 = 1033.50$$

For the second part of the pb (additional information), we have

$$L(\theta) = \left[\prod_{j=1}^{62} f(x_j)\right]\left[S(1000)\right]^{38}$$
$$= \theta^{-62}e^{-28,140/\theta}e^{-38,000/\theta}$$
$$L(\theta)\ = \theta^{-62}e^{-66,140/\theta}$$

$\Rightarrow$

$$l(\theta) = -62\ln(\theta) - 66{,}140 / \theta$$

$$\therefore\ l^{'}(\theta) = \frac{-62}{\theta} + \frac{66{,}140}{\theta^2}$$

$\Rightarrow$

$$-62\theta^2 + 66{,}140\theta = 0$$

$$\therefore\ \hat{\theta} = 1{,}066.77$$

(b)

$$f(x) = \frac{1}{\sqrt{2\pi x}}\exp[-\tfrac{1}{2x}(x-\mu)^2],\ x, \mu > 0$$

$$\Rightarrow \ln f(x|\mu) = \ln\frac{1}{\sqrt{2\pi}} + \ln\frac{1}{\sqrt{x}} - \frac{1}{2x}(x-\mu)^2$$

$$= -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln x - \frac{1}{2}x + \mu - \frac{\mu^2}{2x}$$

$$\because\ l(\mu) = \sum_{j=1}^{n}\ln f_{X_j}(x_j|\mu)$$

$$\therefore\ l(\mu) = -\frac{n}{2}\ln 2\pi - \frac{1}{2}\sum_{j=1}^{n}\ln x_j - \frac{1}{2}\sum_{j=1}^{n}x_j + \mu n - \frac{\mu^2}{2}\sum_{j=1}^{n}\frac{1}{x_j}$$

To get $\hat{\mu}$, set $l^{'}(\mu) = 0$

$$\therefore\ l^{'}(\mu) = n - \mu\sum_{j=1}^{n}\frac{1}{x_j} = 0$$

$$\Rightarrow\ l^{'}(\mu) = n - n\mu y = 0,\ y = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j}$$

$$\therefore\ \hat{\mu} = \frac{1}{y},\ y = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j}$$

$$y = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j}$$

$$\therefore\ \hat{\mu} = \frac{n}{\sum_{j=1}^{n}\frac{1}{x_j}}$$

$$= 20.6774$$