King Saud University
College of Sciences
Department of Mathematics

Final Exam, S3-1444H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 3)

Q1: 8[4+4]

If the random variable X has probability density function $f(x)=(1+2x^2)e^{-2x},\ x\geq 0.$

(a) Determine the survival and hazard rate functions.

(b) Find the mean excess loss function $e_X(x)$ and limited expected value function $E(X\wedge x)$.

Q2: 8[4+4]

(a) Consider the exponential-inverse Gaussian frailty model with

$$a(x)=\frac{\theta}{2\sqrt{1+\theta x}},\ \theta>0$$

(i) Determine the conditional survival function $S_{X|\Lambda}(x|\lambda)$.

(ii) If Λ has a gamma distribution with parameters $\theta=1$ and α replaced by 2α, determine the marginal or unconditional survival function of X.

(b) Claim sizes have an exponential distribution with mean θ. For 80% of risks, $\theta=8$, and for 20% of risks, $\theta=2$. A randomly selected policy had a claim of size 5 in year 1. Determine both the Bayesian and Bühlmann estimates of the expected claim size in year 2.

Q3: 8[4+4]

(a) An insurance company has decided to establish its full-credibility requirements for an individual state rate filing. The full-credibility standard is to be set so that the observed total amount of claims underlying the rate filing would be within 5% of the true value with probability 0.90. The claim frequency follows a Poisson distribution and the severity distribution has pdf

$$f(x)=\frac{100-x}{5{,}000},\quad 0\leq x\leq 100$$

Determine the expected number of claims necessary to obtain full credibility using the normal approximation.

(b) For a particular policyholder, the manual premium is 600 per year. The past claims experience is given in the following table

Year	1	2	3
Claims	475	550	400

Determine the full credibility and partial credibility through premium by assuming the normal approximation. Use $r = 0.05$ and $p = 0.95$.

Q4: 8[2+2+2+2]

The amount of a claim X has an exponential distribution with mean $1/\Theta$. Among the class of insureds and potential insureds, the risk parameter Θ varies according to the gamma distribution with $\alpha = 4$ and scale parameter $\beta = 0.001$. Suppose that a person had claims of 100, 950, and 450.

Find each of the following.

(a) The probability models for X, and risk parameter Θ.

(b) The predictive distribution of the fourth claim and the posterior distribution of Θ.

(c) The Bayesian premium.

(d) The Bühlmann premium.

Hint: For Bühlmann premium use, $\mu = \frac{\beta}{\alpha-1}$, $\upsilon = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$,

where β here is the reciprocal of the usual scale parameter of gamma distribution.

Q5: 8[4+4]

(a) Suppose, you have observed the following five claim severities: 521, 658, 702, 819, and 1,217. Determine the maximum likelihood estimate of α for the following model (which is the Pareto distribution):

$$F(x) = 1 - \left(\frac{500}{x}\right)^{\alpha}, \quad x > 500, \ \alpha > 0.$$

Also, find the value of the log-likelihood function.

(b) Five hundred losses are observed. Five of the losses are 1100, 3200, 3300, 3500, and 3900. All that is known about the other 495 losses is that they exceed 4000. Determine the maximum likelihood estimate of the mean of an exponential model and the value of the log-likelihood function.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: 8[4+4]

(a)

(i)

The survival function is

$$S(x)=\int_x^{\infty}(1+2t^2)e^{-2t}dt$$

$$=-\tfrac{1}{2}e^{-2t}+2I,\ \text{where } I=\int_x^{\infty}t^2e^{-2t}dt$$

$$I=\int t^2e^{-2t}dt$$

$$=-\tfrac{1}{4}e^{-2t}-\tfrac{t}{2}e^{-2t}-\tfrac{t^2}{2}e^{-2t}$$

$$\therefore\ S(x)=-(1+t+t^2)e^{-2t}\Big|_x^{\infty}$$

$$=(1+x+x^2)e^{-2x},\ x\geq 0$$

(ii)

$\because$ The hazard rate function is

$$h(x)=-\frac{d}{dx}[\ln S(x)]$$

and $\because S(x)=(1+x+x^2)e^{-2x}$

$$\ln S(x)=-2x+\ln(1+x+x^2)$$

$$\therefore\ h(x)=2-\frac{1+2x}{1+x+x^2}$$

or simply,

$$h(x)=\frac{f(x)}{S(x)}=\frac{1+2x^2}{1+x+x^2}$$

(b)

(i)

The mean excess loss function is

$$e_X(x) = \frac{\int_x^\infty S(t)dt}{S(x)} \qquad (1)$$

From (i) $S(x) = (1 + x + x^2)e^{-2x}$ (2)

We can deduce that,

$$\begin{aligned}\int_x^\infty S(t)dt &= \int_x^\infty (1 + t + t^2)e^{-2t}dt \\ &= -(1 + t + \tfrac{1}{2}t^2)e^{-2t}\Big|_x^\infty \\ &= (1 + x + \tfrac{1}{2}x^2)e^{-2x} \qquad (3)\end{aligned}$$

Where $I = \int t^2 e^{-2t}dt = -\frac{1}{4}e^{-2t} - \frac{t}{2}e^{-2t} - \frac{t^2}{2}e^{-2t}$,

$\int te^{-2t}dt = -\frac{1}{4}e^{-2t} - \frac{t}{2}e^{-2t}$ and $\int e^{-2t}dt = -\frac{1}{2}e^{-2t}$

$\therefore$ By substituting (2) and (3) in (1), we get

$$e_X(x) = \frac{1 + x + \frac{1}{2}x^2}{1 + x + x^2}$$

(ii)

$$E(X \wedge x) = -\int_{-\infty}^{0} F(t)dt + \int_{0}^{x} S(t)dt$$

$$\begin{aligned}\Rightarrow\ E(X \wedge x) &= 0 + \int_0^x (1 + t + t^2)e^{-2t}dt \\ &= -(1 + t + \tfrac{1}{2}t^2)e^{-2t}\Big|_0^x \\ &= 1 - (1 + x + \tfrac{1}{2}x^2)e^{-2x}\end{aligned}$$

Q2: 8 [4+4]

(a)

(i)

We first find $A(x)$

$$A(x)=\int_0^x a(t)\,dt$$
$$=\int_0^x \frac{\theta}{2\sqrt{1+\theta t}}dt$$
$$=\frac{1}{2}\int_0^x (1+\theta t)^{-\frac{1}{2}}\theta dt$$
$$\therefore A(x)=\sqrt{1+\theta x}-1$$

$$S_{X|\Lambda}(x|\lambda)=e^{-\lambda A(x)}$$
$$=e^{-\lambda(\sqrt{1+\theta x}-1)}$$

(ii)

$\because \Lambda \sim gamma\ (2\alpha,1)$

$\therefore$ The moment generating function of the frailty random variable Λ is

$$M_\Lambda(z)=E(e^{z\Lambda})$$
$$=\left(\frac{1}{1-z}\right)^{2\alpha}=(1-z)^{-2\alpha}$$

The marginal survival function is

$$S_X(x)=E(e^{-\Lambda A(x)})$$
$$=M_\Lambda[-A(x)]$$
$$\therefore S_X(x)=(1+\sqrt{1+\theta x}-1)^{-2\alpha}$$
$$=(1+\theta x)^{-\alpha}$$

Which is a Pareto distribution.

(b)

The Bayesian estimate of the expected claim size in year 2.
We have $\pi(\Theta=8)=0.80$ and $\pi(\Theta=2)=0.20$, and # of claims (claim size) is 5 in year 1.

$$E(X_2|X_1=5)=E(\Theta|X_1=5)$$
$$=\mu(\Theta{=}8)\pi(\Theta=8|X_1=5)+\mu(\Theta{=}2)\pi(\Theta=2|X_1=5)$$
$$\pi(\Theta=8|X_1=5)=\frac{\Pr(X_1=5|\Theta=8)\pi(\Theta=8)}{\Pr(X_1=5|\Theta=8)\pi(\Theta=8)+\Pr(X_1=5|\Theta=2)\pi(\Theta=2)}$$
$$=\frac{(1/8)e^{-5/8}(0.8)}{(1/8)e^{-5/8}(0.8)+(1/2)e^{-5/2}(0.2)}=0.867035$$

Similarly, $\pi(\Theta=2|X_1=5)=\dfrac{\Pr(X_1=5|\Theta=2)\pi(\Theta=2)}{\Pr(X_1=5|\Theta=8)\pi(\Theta=8)+\Pr(X_1=5|\Theta=2)\pi(\Theta=2)}$

$$=\frac{(1/2)e^{-5/2}(0.2)}{(1/8)e^{-5/8}(0.8)+(1/2)e^{-5/2}(0.2)}=0.132965$$

$$\therefore E(X_2|X_1=5)=8\times 0.867035+2\times 0.132965$$

$$=7.2022$$

The Bühlmann estimate of the expected claim size in year 2.

To determine the Bühlmann credibility estimate, we should find the following quantities.

$$\mu=E[\mu(\Theta)]$$
$$=8(0.80)+(2)(0.20)=6.8,$$

$$a=\text{var}[\mu(\Theta)]$$
$$=8^2(0.8)+2^2(0.2)\text{-}6.8^2=5.76,$$

$$v=E[v(\Theta)]$$
$$=\sum_{\theta}v(\theta)\pi(\theta)$$
$$=8^2\times 0.8+2^2\times 0.2=52,$$

Note that for $X\sim\exp(\theta)$ the mean $=E(X)=\theta$ and $\text{var}(X)=\theta^2$

$$k=\frac{v}{a}$$
$$=\frac{52}{5.76}=9.02778,$$

$$Z=\frac{n}{n+k}$$
$$=\frac{1}{1+9.02778}$$
$$\therefore \text{Z}=0.099723.$$

The Bühlmann estimate is

$$E(X_2|100)=P_c=Z\overline{X}+(1-Z)\mu$$
$$=0.099723\times 5+(1-0.099723)\times 6.8$$
$$=6.6205.$$

Q3: 8 [4+4]

(a)

$$\text{at } p = 0.90,\ \Phi(y_p) = (1+p)/2 = 0.95$$
$$\Rightarrow y_p = 1.645 \text{ (by using SND table)}$$
$$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.645/0.05)^2 = 1082.41$$

$$E(X) = \int_0^{100} x\left(\frac{100-x}{5000}\right) dx$$
$$= \int_0^{100} \frac{100x - x^2}{5,000}\, dx$$
$$= \frac{1}{5,000}\left[100(x^2/2) - x^3/3\right]_0^{100}$$
$$\therefore E(X) = \frac{100^3}{5,000}\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{100}{3}$$
$$E(X^2) = \int_0^{100} x^2\left(\frac{100-x}{5,000}\right) dx$$
$$= \int_0^{100} \frac{100x^2 - x^3}{5,000}\, dx = \frac{5,000}{3}$$
$$\therefore Var(X) = E(X^2) - [E(X)]^2$$
$$= \frac{5,000}{3} - \frac{10,000}{9} = \frac{5,000}{9}$$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1 + (\tfrac{\sigma}{\theta})^2]$$

where $\sigma^2 = \dfrac{5,000}{9}$, $\theta^2 = \dfrac{10,000}{9}$

$\therefore$ The expected # of claims $= 1082.41[1+0.5]$

$= 1623.615$

(b)

$$\text{at } p = 0.95,\ \Phi(y_p) = (1+p)/2 = 0.975$$
$$\Rightarrow y_p = 1.96 \text{ (by using SND table)}$$
$$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.96/0.05)^2 = 1536.64$$

The mean is $\xi = E(X_j) = \dfrac{475 + 550 + 400}{3} = 475,$

variance is $\sigma^2 = \dfrac{\sum_j (x_j - \xi)^2}{n-1} = \dfrac{0^2 + 75^2 + 75^2}{2} = 5625$

For full credibility $n \geq \lambda_0 \left(\frac{\sigma}{\xi}\right)^2$

$\therefore n \geq 1536.64\left(\frac{5625}{475^2}\right)$

$\therefore n \geq 38.3095845$

The credibility factor is $Z = \sqrt{\frac{n}{\lambda_0 \sigma^2 / \xi^2}}$

$= \sqrt{\frac{3}{38.3095845}} = 0.279838$

The partial credibility through premium is

$$P_c = Z\overline{X} + (1-Z)M$$

$$= 0.279838(475) + (1-0.279838)(600)$$

$$\therefore P_c = 565.02025$$

Q4: 8 [2+2+2+2]

(a)

For claims, $f_{X|\Theta}(x|\theta) = \theta e^{-\theta x}, \quad x, \theta > 0$

and for the risk parameter,

$$\pi_\Theta(\theta) = \frac{\theta^4 (1000)^4 e^{-\theta/0.001}}{\theta\,\Gamma(4)}, \qquad \theta > 0$$

$$\therefore \pi_\Theta(\theta) = \frac{\theta^3 e^{-1000\theta}(1000)^4}{6}, \qquad \theta > 0$$

(b)

The marginal density at the observed values is

$$f_X(x) = \int \left[\prod_{j=1}^{n} f_{X_j|\Theta}(x_j|\theta)\right]\pi(\theta)d\theta$$

$$f(100, 950, 450) = \int_0^\infty \theta e^{-100\theta} \theta e^{-950\theta} \theta e^{-450\theta}\ \frac{1000^4}{6} \theta^3 e^{-1000\theta} d\theta$$

$$f(100, 950, 450) = \frac{1000^4}{6} \int_0^\infty \theta^6 e^{-2500\theta} d\theta$$

Let $t = 2500\theta$

$$f(100,950,450) = \frac{1000^4}{6}\int_0^\infty \frac{t^6}{(2500)^6} e^{-t} \frac{dt}{2500}$$

$$f(100,950,450) = \frac{1000^4}{6(2500)^7}\int_0^\infty t^6 e^{-t} dt$$

$$\therefore f(100,950,450) = \frac{1000^4}{6(2500)^7}\Gamma(7)$$

$$\therefore f(100,950,450) = \frac{1000^4}{6}\frac{720}{(2500)^7}$$

Similarly,

$$f(100,950,450,x_4) = \int_0^\infty \theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \theta e^{-x_4\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta} d\theta$$

$$= \frac{1000^4}{6}\frac{\Gamma(8)}{(2500+x_4)^8}$$

$$= \frac{1000^4}{6}\frac{5040}{(2500+x_4)^8}$$

The predictive density is

$$f(x_4 \mid 100,950,450) = \frac{f(100,950,450,x_4)}{f(100,950,450)}$$

i.e.

$$f(x_4 \mid 100,950,450) = \frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density with parameters 7 and 2500.

To get posterior density of Θ given X, use the formula

$$\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x}) = \frac{f_{\mathrm{X},\Theta}(\mathrm{x},\theta)}{f_{\mathrm{X}}(\mathrm{x})}$$

$$\therefore \pi(\theta|100,950,450) = \frac{\theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\frac{1000^4}{6}\theta^3 e^{-1000\theta}}{\frac{1000^4}{6}\frac{720}{(2500)^7}}$$

$$= \frac{\theta^6 e^{-2500\theta}(2500)^7}{720}$$

(c) For Bayesian premium,

Since, the amount of a claim has an exponential distribution with mean $1/\Theta$.

$$\therefore \mu_4(\theta) = \theta^{-1} = \frac{1}{\theta}$$

For Bayesian premium estimate, we can use the following Eq.

$$\mathrm{E}(X_{n+1}|\mathrm{X}=\mathrm{x}) = \int \mu_{n+1}(\theta)\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x})d\theta$$

$$\therefore \mathrm{E}(X_4|100,950,450) = \int_0^\infty \frac{1}{\theta}\cdot\frac{\theta^6 e^{-2500\theta}(2500)^7}{720}d\theta$$

$$= \frac{(2500)^7}{720}\int_0^\infty \theta^5 e^{-2500\theta}d\theta$$

$$= \frac{2500}{720}\int_0^\infty u^5 e^{-u}du,\ \ u = 2500\theta$$

$$= \frac{2500}{720}\Gamma(6) = \frac{2500(120)}{720}$$

$$\therefore \mathrm{E}(X_4|100,950,450) = 416.67$$

Another solution

$$\because f(x_4|100,950,450) = \frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density function with parameters $(\alpha,\theta) = (7,2500)$

$$\therefore E(x_4|100,950,450) = \frac{\theta}{\alpha-1} = \frac{2500}{6} = 416,67.$$

(d) To get the Bühlmann premium,

The hypothetical mean is $\mu(\Theta) = \Theta^{-1}$ where $\Theta \sim gamma(4,1000)$, $\alpha = 4$ and $\beta = 1000$ (the reciprocal of the usual scale parameter).

$\mu = E[\mu(\Theta)] = E(\Theta^{-1}) = \dfrac{\beta}{\alpha - 1} = \dfrac{1{,}000}{3}$ is the expected value of hypothetical means,

$\upsilon = E[\upsilon(\Theta)] = E(\Theta^{-2}) = \dfrac{\beta^2}{(\alpha-1)(\alpha-2)} = \dfrac{500{,}000}{3}$ is the expected value of process variance and

$a = Var(\Theta^{-1}) = \dfrac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \dfrac{500{,}000}{9}$ is the variance of hypothetical means.

We can also find a as $a = Var(\Theta^{-1}) = \dfrac{500{,}000}{3} - \left(\dfrac{1{,}000}{3}\right)^2 = \dfrac{500{,}000}{9}$

$$\therefore k = \frac{\upsilon}{a} = 3,\ \text{Z}=\frac{n}{n+k} = \frac{3}{6} = \frac{1}{2}.$$

So, the Bühlmann Premium is

$$P_c = Z\overline{X} + \left(1 - Z\right)\mu$$

$$\therefore P_c = \frac{1}{2}(500) + \frac{1}{2}(\frac{1000}{3}) = 416.67,\quad \text{where } \overline{X} = \frac{100+950+450}{3} = 500,$$

which matches the result of Bayesian premium that introduced in (c).

Q5: 8 [4+4]

(a) $F(x) = 1 - \left(\dfrac{500}{x}\right)^{\alpha}, \quad x > 500,\ \alpha > 0.$

i.e. $X \sim$ single-parameter Pareto (α, θ), where $\theta = 500$

$$\therefore f(x) = \frac{\alpha (500)^{\alpha}}{x^{\alpha+1}}$$

The log-likelihood function is

$$\therefore \ln f(x|\alpha) = \ln\alpha + \alpha \ln 500 - (\alpha+1)\ln x$$

$$\because l(\alpha) = \sum_{j=1}^{n} \ln f_{X_j}(x_j|\alpha)$$

$$\therefore l(\alpha) = \sum_{j=1}^{5} (\ln\alpha + \alpha \ln 500 - (\alpha+1)\ln x_j)$$

$$= 5\ln\alpha + 5\alpha\ln 500 - (\alpha+1)\sum_{j=1}^{5}\ln x_j$$

To get $\hat{\alpha}$, set $l'(\alpha) = 0$

$$\Rightarrow \frac{5}{\alpha} + 5\ln 500 - \sum_{j=1}^{5}\ln x_j = 0$$

$$\therefore 5\alpha^{-1} + 5\ln 500 - 33.1111 = 0$$

$$\Rightarrow 5\alpha^{-1} - 2.0381 = 0$$

$$\therefore \hat{\alpha} = \frac{5}{2.0381} \simeq 2.45$$

$$\therefore l(\hat{\alpha}) \simeq -33.6239$$

(b)

$$L(\theta) = f(1100)f(3200)f(3300)f(3500)f(3900).[S(4000)]^{495}$$

$$= \theta^{-1}e^{-1100/\theta}\theta^{-1}e^{-3200/\theta}\theta^{-1}e^{-3300/\theta}\theta^{-1}e^{-3500/\theta}\theta^{-1}e^{-3900/\theta}[e^{-4000/\theta}]^{495}$$

$$= \theta^{-5}e^{-1995000/\theta}$$

$$\Rightarrow l(\theta) = -5\ln\theta - \frac{1995000}{\theta}$$

To get $\hat{\theta}$, set $l'(\theta) = 0$

$$\therefore \frac{\partial l}{\partial\theta} = \frac{-5}{\theta} + \frac{1995000}{\theta^2} = 0$$

$$\Rightarrow \hat{\theta} = \frac{1995000}{5} = 399000$$

and $l(\hat{\theta}) = -69.4836$