King Saud University
College of Sciences
Department of Mathematics

Midterm Exam, S1-1444H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 minutes - Marks: 30

Answer the following questions.

Q1: [5+5]

Determine the mean excess loss and limited expected value functions for the model of the age at death that defined as:

$$F(x)=\begin{cases} 0, & x<0 \\ 0.01x, & 0\le x<100 \\ 1, & x\ge 100 \end{cases}$$

Q2: [5+5]

(a) Let Λ have a gamma distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda)=e^{-\lambda x^{\gamma}}$. Determine the unconditional or marginal distribution of X.

(b) Show that the Weibull distribution has a scale parameter.

Q3: [5+5]

(a) Using the following data set,

Payment range	0-7500	7500-17500	17,500-32,500	32,500-67,500	67,500-125,000	125,000-300,000	Over 300,000
Number of payments	99	42	29	28	17	9	3

Determine the loglikelihood function.

(b) Let X have a Pareto distribution with parameters α and θ. Let $Y=\ln(1+X/\theta)$. Determine the name of the distribution of Y and its parameters.

The Model Answer

Q1: [5+5]

(i)

$$\because e_X(d) = \frac{\int_d^{\infty} S(x)dx}{S(d)}$$

$$= \frac{\int_d^{100} (1-0.01x)dx}{1-0.01d}$$

$$= \frac{-1}{0.01(1-0.01d)}\left[\frac{(1-0.01x)^2}{2}\right]_d^{100}$$

$$\therefore e_X(d) = \frac{1-0.01d}{0.02} = \frac{100-d}{2}$$

(ii)

$$\because E(X \wedge u) = \int_{-\infty}^{u} xf(x)dx + u[1-F(u)]$$

$$= \int_0^u x(0.01)dx + u(1-0.01u)$$

$$= 0.005u^2 + u - 0.01u^2$$

$$\therefore E(X \wedge u) = u(1-0.005u)$$

or

$$E(X \wedge u) = \int_0^u S(x)dx$$

$$= \int_0^u (1-0.01x)dx$$

$$= \left[x - \frac{0.01x^2}{2}\right]_0^u$$

$$\therefore E(X \wedge u) = u - 0.005u^2$$

or

By using the following formula

$$E(X \wedge u) = E(X) - e(u)S(u)$$

$\because E(X) = \int_0^u x(0.01)dx = \int_0^{100} x(0.01)dx$

$= 0.01\left[\frac{x^2}{2}\right]_0^{100}$

$\therefore$ E(x) = 50

Also, $e(u) = \frac{100-u}{2}$ and $S(u) = 1 - 0.01u$

$\because E(X \wedge u) = E(X) - e(u)S(u)$

$= 50 - (\frac{100-u}{2})(1 - 0.01u)$

$= 50 - \frac{0.01}{2}(100-u)^2$

$\therefore E(X \wedge u) = u - 0.005u^2$

Q2: [5+5]

(a)

let Λ ~ gamma (θ,α), $X|\Lambda$ ~ weibull(λ,γ)

$\therefore \; S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$

$\therefore \; A(x) = x^{\gamma}$

$\because \; S_X(x) = E[e^{-\Lambda A(x)}]$

$= M_{\Lambda}[-A(x)]$

$\therefore \; S_X(x) = M_{\Lambda}[-x^{\gamma}]$

and $\because M_{\Lambda}(z) = (1-\theta z)^{-\alpha}$

$\therefore \; S_X(x) = (1+\theta x^{\gamma})^{-\alpha}$

which is a Burr distribution with parameters

$\theta \to \theta^{\frac{-1}{\gamma}}, \; \alpha \to \alpha$

(b)

For two parameter Weibull distribution - $\theta, \; \tau$

$F_X(x) = 1 - e^{-(x/\theta)^{\tau}}$

Let $Y = cX$, where $c > 0$ Then

$$F_Y(y) = \Pr(cX \le y) = \Pr(X \le \tfrac{y}{c})$$
$$\therefore F_Y(y) = 1 - e^{-(y/c\theta)^\tau}$$

which is also a Weibull distribution, with parameters τ and $c\theta$.

$\therefore \theta$ is a scale parameter.

Q3: [5+5]

(a) The loglikelihood function is

$$\begin{aligned} l(\theta) = {} & 99\ln[F(7500) - F(0)] + 42\ln[F(17,500) - F(7500)] \\ & + 29\ln[F(32,500) - F(17,500)] + 28\ln[F(67,500) - F(32,500)] \\ & +17\ln[F(125,000) - F(67,500)] + 9\ln[F(300,000) - F(125,000)] \\ & +3\ln[(1 - F(300,000)] \end{aligned}$$

$$\begin{aligned} l(\theta) = {} & 99\ln[1 - \exp(-7,500/\theta)] + 42\ln[\exp(-7,500/\theta) - \exp(-17,500/\theta)] \\ & + 29\ln[\exp(-17,500/\theta) - \exp(-32,500/\theta)] + 28\ln[\exp(-32,500/\theta) - \exp(-67,500/\theta)] \\ & +17\ln[\exp(-67,500/\theta) - \exp(-125,000/\theta)] + 9\ln[\exp(-125,000/\theta) - \exp(-300,000/\theta)] \\ & +3\ln[\exp(-300,000/\theta)] \end{aligned}$$

(b)

$$\because X \sim Pareto\ (\alpha, \theta)$$
$$\therefore F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha$$

For $Y = \ln(1 + X/\theta)$, $F_Y(y) = \Pr(Y \le y)$

$$F_Y(y) = \Pr[\ln(1 + X/\theta) \le y]$$

$$\begin{aligned} \therefore F_Y(y) &= \Pr[1 + X/\theta \le e^y] \\ &= \Pr[X \le \theta(e^y - 1)] \\ &= 1 - \left[\frac{\theta}{\theta(e^y - 1) + \theta}\right]^\alpha \end{aligned}$$

$$\begin{aligned} \therefore F_Y(y) &= 1 - \left(\frac{1}{e^y}\right)^\alpha \\ &= 1 - e^{-\alpha y}, \end{aligned}$$

which is the distribution function of the exponential distribution with parameter $1/\alpha$.